5. Cosmic Expansion History

Reading: Chapter 5, don't get too lost in the equations

Solutions of the Friedmann Equation

We now understand the origin of the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2}\frac{1}{a^2(t)},$$

and the dependence of energy density $\epsilon(t)$ on the expansion factor $a(t)$ for various forms of matter and energy.
Now we want to know what the *solutions* of the Friedmann equation are for interesting cases.

For single-component universes, the following cases can be derived by integrating the Friedmann equation or verified by substitution.

Matter-dominated, critical density ($k = 0$)

$$\epsilon = \epsilon_0 (a/a_0)^{-3}, \qquad a(t) = (t/t_0)^{2/3}, \qquad H = 2/3t.$$

Radiation-dominated, critical density ($k = 0$)

$$\epsilon = \epsilon_0 (a/a_0)^{-4}, \qquad a(t) = (t/t_0)^{1/2}, \qquad H = 1/2t.$$

Empty ($k = -1$)

$$\epsilon = 0, \qquad a(t) = (t/t_0), \qquad H = 1/t.$$

Cosmological constant, critical density ($k = 0$)

$$\epsilon = \epsilon_\Lambda = \frac{3H_0^2 c^2}{8\pi G}, \qquad a(t) = e^{H_0(t-t_0)}.$$

For the $H(t)$ relation in the matter dominated case, note that if $a \propto t^{2/3}$ then

$$\frac{d\ln a}{d\ln t} = \frac{t}{a}\frac{da}{dt} = tH = \frac{2}{3}$$

and thus $H = 2/3t$. The same argument works for the radiation-dominated and empty cases.

Matter-dominated with curvature

Matter-dominated solutions with curvature are more complicated, but the solutions can be expressed in parametric form (textbook equations 5.90 and 5.91 for $k = +1$, and 5.93 and 5.94 for $k = -1$).

For $k = +1$, the universe expands, reaches a maximum at

$$a_{\max} = \frac{\Omega_0}{\Omega_0 - 1},$$

then recollapses and ends in a "big crunch."

For $k = -1$, the universe expands forever.
From the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2}\frac{1}{a^2(t)},$$

we can see that the two terms on the right-hand side are equal when $\Omega(t) = 0.5$.
At much earlier times, the first term must dominate, and the universe should evolve like a critical density universe.
At much later times, the second term must dominate, and the universe should evolve like an empty universe.

Specifically, when $a \ll \Omega_0/(1-\Omega_0)$, the expansion is very close to $a(t) \propto t^{2/3}$.
When $a \gg \Omega_0/(1-\Omega_0)$, the expansion is very close to $a(t) \propto t$, i.e., "free expansion" (not accelerating or decelerating).

A flat universe with matter and a cosmological constant

A flat universe dominated by matter and a cosmological constant appears to be a good description of the cosmos we live in.
The behavior is analogous to that of the $k = -1$ matter-dominated solution discussed above, except that the transition is to exponential expansion rather than free expansion.

The Friedmann equation in this case can be written

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \Omega_{\Lambda,0}\right].$$

(Since $a_0 \equiv 1$, we could just drop it from this equation.)
The transition occurs at an expansion factor

$$\left(\frac{a_{m\Lambda}}{a_0}\right) \sim \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}.$$

Note that for a critical density universe, we must have $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$.

Matter-radiation equality

The energy density of radiation in the universe is $\epsilon_{r,0} = a_{\mathrm{SB}}T^4$, where $T = 2.7\,\mathrm{K}$ is the temperature of the cosmic microwave background (CMB).
Roughly, CMB photons have $\lambda \sim 1\,\mathrm{mm}$, $kT \sim 10^{-3}\mathrm{eV}$, $n_\gamma \sim 10^3\,\mathrm{photons\,cm^{-3}}$, implying $\epsilon_{r,0} \sim 1\mathrm{eV\,cm^{-3}}$.
The contribution of starlight is negligible compared to the CMB.

The mean density of hydrogen atoms is roughly one atom per cubic meter. Hence there are about 10^9 CMB photons per baryon, and $\epsilon_{\mathrm{bary},0} \sim 10^9\mathrm{eV}/10^6\,\mathrm{cm}^3 \sim 10^3\mathrm{eV\,cm^{-3}} \sim 10^3\epsilon_{r,0}$.
(With more accurate numbers, the mean density of hydrogen atoms is about 0.2 per cubic meter, and the ratio of CMB photons to baryons is about 2×10^9.)

Since $\epsilon_{\rm bary} = \epsilon_{\rm bary,0}(a_0/a)^3 = \epsilon_{\rm bary,0}(1+z)^3$, and $\epsilon_r = \epsilon_{r,0}(a_0/a)^4 = \epsilon_{r,0}(1+z)^4$, the radiation and baryon densities were equal at

$$(1+z) \sim \frac{\epsilon_{\rm bary,0}}{\epsilon_{r,0}} \sim 1000.$$

The "Benchmark Model"

To be more precise, we need to use precise numbers for the above *and* include two other important contributions.

The radiation component includes neutrinos, which are nearly as numerous as CMB photons and are highly relativistic in the early universe.

The matter component includes dark matter, which appears to outweigh baryons by a factor $\sim 6:1$.

The Benchmark Model in Table 5.2 of the textbook has good current numbers, derived principally from cosmic microwave background anisotropy measurements assuming a flat universe with cold dark matter and a cosmological constant.

For $H_0 = 68 {\rm km\ s^{-1}\ Mpc^{-1}}$,

$$\begin{aligned}
&\Omega_{\gamma,0} = 5.35 \times 10^{-5} \\
&\Omega_{\nu,0} = 3.65 \times 10^{-5} \\
&\Omega_{r,0} = \Omega_{\gamma,0} + \Omega_{\nu,0} = 9.0 \times 10^{-5} \\
&\Omega_{\rm bary,0} = 0.048 \\
&\Omega_{\rm DM,0} \approx 0.262 \\
&\Omega_{m,0} = \Omega_{\rm bary,0} + \Omega_{\rm DM,0} \approx 0.31 \\
&\Omega_{\Lambda,0} = 1 - \Omega_{m,0} \approx 0.69.
\end{aligned}$$

With these numbers, $t_0 = 0.96/H_0 = 13.7\,{\rm Gyr}$.

The value of $\Omega_{\gamma,0}$ is precisely known from the CMB temperature.

The cosmic neutrino background cannot be measured directly, but the value of $\Omega_{\nu,0}$ can be precisely calculated from theory given the standard model of particle physics. However, we have here treated neutrinos as *massless*, which is an excellent approximation in the early universe but not today.

The value of $\Omega_{\rm bary,0}$ is well determined (at the $\approx 10\%$ level) by measurements of the cosmic deuterium abundance and by measurements of anisotropy in the CMB.

The value of $\Omega_{\rm DM,0}$ is the most uncertain, but a variety of measurements imply it is probably known to 10%.

Based on these numbers, we conclude that radiation and matter had equal energy density at redshift

$$(1+z_{\rm eq}) = \frac{\Omega_{m,0}}{\Omega_{r,0}} \approx 3450.$$

At redshifts much higher than $z_{\rm eq}$, the universe was radiation dominated, with $a(t) \propto t^{1/2}$.

At $z_{\rm eq}$ there was a transition to a matter dominated universe, with $a(t) \propto t^{2/3}$.

At much lower redshift ($z \lesssim 2$), dark energy became important.

The age of the universe at $z_{\rm eq}$ is $t_{\rm eq} = 4.7 \times 10^4$ yrs.

Another Form of the Friedmann Equation

Using the definitions of Ω and $H = \dot{a}/a$, one can write the Friedmann equation in another useful form (textbook eq. 5.81):

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \ ,$$

where $\Omega_0 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}$ is equal to 1 for a flat universe.

If you know H_0 and the present-day values of the density parameters, you can use this form to compute the expansion rate $H(t)$ at earlier times.

The definition $H = \dot{a}/a$ implies $dt = da/aH$. This can be combined with the above equation to get an expression (textbook eq. 5.83) for the age of the universe at redshift z, expansion factor $a = (1+z)^{-1}$.

$$t = \int_0^a dt = \int_0^1 \frac{da}{aH} = H_0^{-1} \int_0^a \frac{da}{\left[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0} a^2 + (1-\Omega_0)\right]^{1/2}} \ .$$

We can also calculate the comoving distance to an object at redshift z:

$$r = \int_a^1 dr = \int_a^1 \frac{c\,dt}{a(t)} = \int_a^1 \frac{c\,da}{a^2 H} = cH_0^{-1} \int_a^1 \frac{da}{\left[\Omega_{r,0}/a^4 + \Omega_{m,0}/a^3 + \Omega_{\Lambda,0} + (1-\Omega_0)/a^2\right]^{1/2}} \ .$$

These expressions can also be written with z as the integration variable using $da = -(1+z)^{-2} dz$.

For general combinations of $\Omega_{x,0}$ these integrals must be done numerically, though they can be done analytically in some special cases.