Astronomy 5682
Problem Set 3: Measuring Curvature, Calculating Expansion
Due Thursday, Feb 16, 4 pm

Background Reading for Questions 1 and 2: Ryden §§3.4-3.6. You may also find §6.3 useful for Question 2.

Background reading for Question 3: Chapter 5, §§5.1-5.3

Note that you have two weeks for this assignment. I recommend starting early.

You may turn this assignment in on paper, either in-class on Wednesday (2/15) or to my mailbox in 4055 McPherson on Thursday. Alternatively, you may submit it electronically via Carmen.

Late assignments will be accepted until 4 pm Monday (2/20) with a 10-point penalty (out of 100).

If you need to make any special arrangements for turning in the assignment please e-mail *both* me (weinberg.21@osu.edu) and GTA Joy Bhattacharyya (bhattacharyya.37@osu.edu).

Question 1: Angular Sizes in an Expanding Universe (30 points)

Recall that:

- The FRW metric is
$$ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + S_k^2(r)d\Omega^2\right],$$
where
$$\begin{aligned} S_k(r) &= R_0 \sin(r/R_0), \qquad k = +1, \\ &= r, \qquad\qquad\qquad k = 0, \\ &= R_0 \sinh(r/R_0), \qquad k = -1. \end{aligned} \tag{2}$$
- The comoving distance to an object that emitted light at time t_e is
$$r = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} \tag{3}$$
- The cosmological redshift is $1 + z = \frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e}$, and by definition $a_0 = a(t_0) = 1$.

(a) Consider a universe that has $a(t) = (t/t_0)^{2/3}$, flat space geometry ($k = 0$), and $t_0 = 14$ Gyr (14 billion years). *What is* the comoving distance r to a galaxy at redshift $z = 3$? Express your answer in light years.

(b) Suppose this galaxy is oriented perpendicular to the line of sight and subtends an angle $d\Omega = 3.6$ arc-seconds (10^{-3} degrees). *What is* the galaxy's physical size, in light years?

(c) Consider instead a universe that has $a(t) = t/t_0$, negative space curvature ($k = -1$), a curvature radius $R_0 = ct_0$, and $t_0 = 14$ Gyr. *What is* the comoving distance r to a galaxy at $z = 3$?

(d) Suppose this galaxy is oriented perpendicular to the line of sight and subtends an angle $d\Omega = 3.6$ arc-seconds (10^{-3} degrees). *What is* the galaxy's physical size, in light years?

(e) Suppose you observed an object (maybe a galaxy, maybe something else) of known physical size and measured its redshift, $z = 3$. *How could you use* the observed angular size of the object to decide which of the two models (the one in a/b or the one in c/d) better describes the real universe? *In which kind of universe* would the object have the larger angular size?

Question 2: Constraining Cosmic Curvature (40 points)

Galaxies do not come with their true physical sizes conveniently labeled, so the test of 2(e) cannot be carried out in practice. However, there *is* a characteristic scale imprinted on the *clustering* of galaxies in space known as the baryon acoustic oscillation (BAO) feature, which will be discussed in Paul Martini's guest lecture. The size of the BAO feature can be calculated from known physics, and for this problem we'll take it to be exactly 500 million light years in *comoving* units — i.e., the physical size at redshift z is $500(1+z)^{-1}$ million light years.

Suppose that you measure the angular scale of the BAO feature from the clustering of galaxies at redshift $z = 0.5$ and find a result that is consistent with the expectation for a flat ($k = 0$) universe, with a measurement uncertainty of 1%. (BOSS, The Baryon Oscillation Spectroscopic Survey, has actually done this.)

(a) *Show that* in a universe with $a(t) = (t/t_0)^{2/3}$ the comoving distance to $z = 0.5$ is $r = 0.551ct_0$.

(b) Suppose that the value $t_0 = 14\,\mathrm{Gyr}$ is known exactly, and that $k = 0$. *What is* the expected angular size corresponding to the 500 million-light-year BAO scale? Give your answer in degrees.

(c) Your measurement is consistent with this prediction, but with 1% uncertainty. *What lower limit can you set* on the value of the curvature radius R_0, assuming that this is the only source of uncertainty in the problem?

Hint: See the example solution provided for textbook question 3.3.

(d) You decide to carry out a larger survey to improve the precision of your measurement, and you have two choices: a survey that will yield a 0.5% measurement at $z = 0.5$ or a survey that will yield a 1% measurement at $z = 1$. *Which of these* will give you a better constraint on curvature (i.e., a stronger lower limit on R_0 assuming the measurement is still compatible with flatness)?

The Dark Energy Spectroscopic Instrument (DESI) project is conducting a giant redshift survey of about 30 million galaxies to measure BAO distances with precision of about 0.2%, out to a redshift $z \approx 1.5$. OSU has played a big role in constructing the spectrographs for DESI, which use optical fibers to observe 5000 objects at a time. The survey will begin about 2 years ago and will run for 5 years total, on the 4-meter Mayall telescope at Kitt Peak. Its primary objective is to learn about dark energy by measuring the expansion history $a(t)$.

Question 3: A Matter-Dominated Universe (30 points)

As discussed in class and in Chapter 4 of the textbook, the Friedmann Equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2}\frac{1}{a^2(t)}.$$

(a) *Explain why* a universe with present energy density

$$\frac{\epsilon_0}{c^2} = \frac{3H_0^2}{8\pi G}$$

must have $k = 0$.

The quantity $\rho_{\mathrm{crit}} = 3H_0^2/(8\pi G)$ is referred to as the critical density.

(b) *Argue that* if the dominant form of energy in the universe is non-relativistic matter (i.e., atoms or particles whose kinetic energy is negligible compared to their rest mass energy) then

$$\epsilon(t) = \epsilon_0 \left[a(t)\right]^{-3}.$$

(c) *Show that* in a critical density, matter-dominated universe, the evolution of the expansion factor is given by

$$a(t) = (t/t_0)^{2/3}.$$

(d) *Show that* the age of a critical density, matter-dominated universe is

$$t_0 = \frac{2}{3} \times \frac{1}{H_0}.$$

[*Hint:* Recall that $H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt}$, and $t_0 = \int_{a=0}^{a=1} dt$.]

(e) *Show that* the comoving distance to an object at redshift z in a critical density, matter dominated universe is

$$r = \frac{2c}{H_0}\left[1 - a_e^{1/2}\right],$$

where $a_e = (1+z)^{-1}$ is the expansion factor at the time the object emitted its light.