4 Pressure and Viscosity

Reading: Ryden, chapter 2; Shu, chapter 4

4.1 Specific heats and the adiabatic index

First law of thermodynamics (energy conservation):

$$d\epsilon = -PdV + dq \quad \Longrightarrow \quad dq = d\epsilon + PdV, \tag{28}$$

$$
\begin{aligned}
V &\equiv \rho^{-1} = \text{specific volume} \quad [\,\mathrm{cm^3\,g^{-1}}\,] \\
dq &\equiv Tds = \text{heat change per unit mass} \quad [\,\mathrm{erg\,g^{-1}}\,] \\
s &\equiv \text{specific entropy} \quad [\,\mathrm{erg\,g^{-1}\,K^{-1}}\,].
\end{aligned}
$$

The specific heat at constant volume,

$$c_V \equiv \left(\frac{\partial q}{\partial T}\right)_V \quad [\,\mathrm{erg\,g^{-1}\,K^{-1}}\,] \tag{29}$$

is the amount of heat that must be added to raise temperature of $1\,\mathrm{g}$ of gas by $1\,\mathrm{K}$.

At constant volume, $dq = d\epsilon$, and if ϵ depends only on temperature (not density), $\epsilon(V,T) = \epsilon(T)$, then

$$c_V \equiv \left(\frac{\partial q}{\partial T}\right)_V = \left(\frac{\partial \epsilon}{\partial T}\right)_V = \frac{d\epsilon}{dT}.$$

implying

$$dq = c_V dT + PdV.$$

If a gas has temperature T, then each degree of freedom that can be excited has energy $\frac{1}{2}kT$. (This is the equipartition theorem of classical statistical mechanics.)
The pressure

$$P = \frac{1}{3}\rho\langle|\vec{w}|^2\rangle = \frac{\rho}{m}kT$$

since

$$\langle\frac{1}{2}mw_i^2\rangle = \frac{1}{2}kT \quad \Longrightarrow \quad \langle|\vec{w}|^2\rangle = \frac{3kT}{m}.$$

Therefore

$$PV = \frac{kT}{m} \quad \Longrightarrow \quad PdV = \frac{k}{m}dT.$$

Using $dq = c_V dT + PdV$, the specific heat at constant pressure is

$$c_P \equiv \left(\frac{\partial q}{\partial T}\right)_P = c_V + P\frac{dV}{dT} = c_V + \frac{k}{m}.$$

Changing the temperature at constant pressure requires more heat than at constant volume because some of the energy goes into PdV work.

For reasons that will soon become evident, the quantity $\gamma \equiv c_P/c_V$ is called the adiabatic index. A monatomic gas has 3 degrees of freedom (translation), so

$$\epsilon = \frac{3}{2}\frac{kT}{m} \implies c_V = \frac{3}{2}\frac{k}{m} \implies c_P = \frac{5}{2}\frac{k}{m} \implies \gamma = \frac{5}{3}.$$

A diatomic gas has 2 additional degrees of freedom (rotation), so $c_V = 5k/2m$, $\gamma = 7/5$. More generally

$$\epsilon = \frac{1}{\gamma - 1}\frac{kT}{m} = \frac{1}{\gamma - 1}\frac{P}{\rho}.$$

4.2 Adiabatic evolution

As discussed in §2.7, a perfect gas has a Maxwellian velocity distribution and therefore no viscosity so it obeys the Euler equations.

In the absence of radiative heating and cooling, one can combine the continuity and energy equations,

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{u}, \qquad \frac{D\epsilon}{Dt} = -\frac{P}{\rho}\vec{\nabla}\cdot\vec{u},$$

to find

$$\frac{D\epsilon}{Dt} = \frac{P}{\rho^2}\frac{D\rho}{Dt} = -P\frac{DV}{Dt},$$

and since $d\epsilon = -PdV + Tds$ we conclude that

$$T\frac{Ds}{Dt} = 0.$$

In the absence of radiative heating and cooling, a perfect gas undergoes only adiabatic (constant entropy) changes.

From the equation of state

$$\epsilon = \frac{1}{\gamma - 1}\frac{P}{\rho}$$

we have

$$d\epsilon = \frac{1}{\gamma - 1}\left(\frac{dP}{\rho} - \frac{P}{\rho^2}d\rho\right).$$

For adiabatic ($ds = 0$) changes, we can combine this with the first law of thermodynamics

$$d\epsilon = -PdV = \frac{P}{\rho^2}d\rho$$

to find (after multiplying by ρ/P)

$$\frac{1}{\gamma - 1}\left(\frac{dP}{P} - \frac{d\rho}{\rho}\right) = \frac{d\rho}{\rho}$$

implying

$$\frac{dP}{P} = \gamma\frac{d\rho}{\rho} \implies P = P_0(\rho/\rho_0)^\gamma,$$

a polytropic equation of state ($P \propto \rho^\gamma$).

4.3   Summary: single particle species equation of state

The thermal pressure is

$$P = \frac{1}{3}\rho\langle|\vec{w}|^2\rangle = \frac{\rho}{m}kT = nkT. \tag{30}$$

The specific internal energy is

$$\epsilon = \frac{1}{\gamma - 1}\frac{kT}{m} = \frac{1}{\gamma - 1}\frac{P}{\rho}, \tag{31}$$

where $\gamma = c_P/c_V$ is the adiabatic index.
A gas undergoing only adiabatic changes has a polytropic equation of state $P = P_0(\rho/\rho_0)^\gamma$.
A change in entropy changes the "adiabat" of the gas, i.e., the relation between P_0 and ρ_0.

4.4   Gas entropy

Start with

$$d\epsilon = -PdV + Tds$$

and consider adding or removing heat at constant ρ $(dV = 0)$

$$Tds = d\epsilon = c_V dT$$

implying

$$ds = c_V \frac{dT}{T} \qquad \Longrightarrow \qquad s = c_V \ln T + \text{const.}.$$

Since $P \propto T$ at constant ρ, this implies that $s = c_V \ln P + \text{const.}$.

What about changes of density? We know that adiabatic changes keep $P\rho^{-\gamma}$ constant, so these must be lines of constant entropy (in the plane of pressure and density).

Therefore, for a single particle species gas,

$$s = c_V \ln\left(P\rho^{-\gamma}\right) + \text{const.} \tag{32}$$

4.5   Cluster scaling relations and the "entropy floor"

Consider a simple model of galaxy clusters in which the density profile of the hot intracluster gas is $\rho(r) = \rho_V(R/r)^2$ for $r > r_0$ and $\rho(r) = \text{const.}$ for $r <= r_0$. Define the cluster concentration parameter $c = R/r_0$, implying $\rho_0 = c^2\rho_V$.

For Bremmstrahlung (free-free) emission, the emissivity per unit volume is proportional to $\rho^2 T^{1/2}$. The cluster's X-ray luminosity will be dominated by the high density core, and we therefore expect

$$L_x \sim \rho_0^2 T^{1/2} r_0^3 \sim \rho_V^2 c^4 T^{1/2} c^{-3} R^3 \sim \rho_V^2 c R^3 T^{1/2}.$$

Now make two fairly general assumptions about the cluster population, that they follow a virial relation $T \sim GM/R$ and that they all have roughly the same average density $M/R^3 \sim \rho_V = \text{const.}$ (indeed, one can *define* the cluster radius to be the radius where the density falls to ρ_V.)

Under these assumptions, we find

$$R \sim M^{1/3} \quad \Longrightarrow \quad T \sim M^{2/3} \sim R^2$$

and

$$L_x \sim cR^3 T^{1/2} \sim cT^2 \sim cM^{4/3}.$$

If all clusters have similar profile shapes, so that c is the same for all clusters, we therefore expect $L_x \propto T^2 \propto M^{4/3}$.

However, observed clusters show an $L_x - T$ relation that is more like $L_x \propto T^3$, perhaps even steeper in the regime of low mass groups. One popular explanation of this discrepancy is that gas was "pre-heated" by supernova winds or some other feedback mechanism before falling into clusters, giving it an "entropy floor" — a minimum level $s_{\min}$ below which it cannot fall.

The cluster's central entropy is $s_0 \sim T\rho_0^{-2/3} \sim Tc^{-4/3}$. If all clusters have $s_0 = s_{\min} = \text{const.}$, then

$$c \sim T^{3/4} s_{\min}^{-3/4} \quad \Longrightarrow \quad L \sim T^{2.75}.$$

Thus, an entropy floor leads to larger cores (relative to R) in cooler clusters and thus to a steeper $L_x - T$ relation.

Alternative possibilities are that the gas is heated after falling into the clusters (though then more energy must be injected to "puff it up" by a significant amount) or that an approximate entropy "floor" arises because the low entropy gas cools and settles into galaxies, with higher entropy gas flowing in to replace it.

4.6 Multiple particle species and mean molecular weight

Now suppose that we have particle species $j = 1, N$, each with mass m_j.
Define the mean molecular weight

$$\mu \equiv \frac{\rho}{nm_p} = \frac{\sum n_j m_j}{nm_p}, \tag{33}$$

the ratio of the number-weighted mean particle mass to the proton mass, $n = \sum n_j$.

Since $\frac{1}{2}m_j \langle |\vec{w}|^2 \rangle_j = \frac{3}{2}kT$, the pressure and specific internal energy of each species is

$$\begin{aligned} P_j &= \frac{1}{3}\rho_j \langle |\vec{w}|^2 \rangle_j = \frac{\rho_j}{m_j}kT = n_j kT, \\ \epsilon_j &= \frac{1}{2}\langle |\vec{w}|^2 \rangle_j = \frac{3}{2}\frac{kT}{m_j}, \end{aligned}$$

assuming that all species are characterized by the same temperature T.

The total pressure is

$$P = \sum_j P_j = \sum_j n_j kT = nkT = \frac{\rho}{\mu m_p}kT.$$

For the specific internal energy, we must weight according to the mass density in each species,

$$\epsilon = \frac{\sum \epsilon_j m_j n_j}{\sum m_j n_j} = \frac{\sum \frac{3}{2} kT n_j}{\rho} = \frac{3}{2} kT \frac{n}{\rho} = \frac{3}{2} \frac{kT}{\mu m_p} = \frac{3}{2} \frac{P}{\rho}.$$

So the single species formulas apply, with the substitution $m \longrightarrow \mu m_p$.

For primordial composition, helium = 7% by number (23% by mass).

$$\begin{aligned} \text{neutral:} \quad & \mu = \frac{0.93 m_p + 4 \times 0.07 m_p}{(0.93 + 0.07) m_p} && \approx 1.2 \\ \text{ionized:} \quad & \mu = \frac{0.93 m_p + 4 \times 0.07 m_p}{(0.93 + 0.93 + 0.07 + 2 \times 0.07) m_p} && \approx 0.6 \quad (m_e \ll m_p). \end{aligned}$$

4.7 Molecular viscosity

Returning to the Navier-Stokes equation, the momentum conservation equation is

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \vec{g} - \frac{1}{\rho} \vec{\nabla} P + \frac{1}{\rho} \vec{\nabla} \cdot \overset{\leftrightarrow}{\pi},$$

where

$$\vec{\nabla} \cdot \overset{\leftrightarrow}{\pi} = \sum_i \frac{\partial}{\partial x_i} \pi_{ij} \qquad \pi_{ij} = P \delta_{ij} - \rho \langle w_i w_j \rangle.$$

In a "Newtonian fluid," $\overset{\leftrightarrow}{\pi}$ is linearly proportional to the velocity gradient $\frac{\partial u_i}{\partial x_j}$ (this was essentially a guess on the part of Newton and Hooke).

The most general symmetric tensor linear in $\frac{\partial u_i}{\partial x_j}$ is

$$\pi_{ij} = \mu D_{ij} + \beta \delta_{ij} \left(\vec{\nabla} \cdot \vec{u} \right), \tag{34}$$

where the deformation tensor

$$D_{ij} \equiv \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \left(\vec{\nabla} \cdot \vec{u} \right) \tag{35}$$

vanishes for uniform expansion or contraction, and

$$\begin{aligned} \mu \equiv \quad & \text{coefficient of shear viscosity} \quad = [\,\mathrm{g\,cm^{-1}\,s^{-1}}\,] \\ \beta \equiv \quad & \text{coefficient of bulk viscosity} \quad = [\,\mathrm{g\,cm^{-1}\,s^{-1}}\,]. \end{aligned}$$

(Note that this μ has nothing to do with mean molecular weight.)
μD_{ij} represents resistance to shearing motion and $\beta \delta_{ij} \left(\vec{\nabla} \cdot \vec{u} \right)$ represents resistance to changes in volume.

The value of μ can be estimated at an order-of-magnitude level as described in Shu (pp. 30-32), similar to Ryden (pp. 16-17). I have not come across a similar estimate of β.

Consider a plane-parallel flow with shear, $u_y = u_z = 0$, $\frac{\partial u_x}{\partial y} \neq 0$, and focus on a volume $\Delta A \Delta y$ bounded by surfaces of area ΔA separated by Δy.

If we further assume constant density and pressure and ignore gravitational accelerations, the momentum equation becomes (since $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0$)

$$\frac{\partial}{\partial t}(\rho u_x) = \frac{\partial}{\partial y}\pi_{xy} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u_x}{\partial y}\right),$$

so $\frac{\partial}{\partial y}\left(\mu\frac{\partial u_x}{\partial y}\right)$ is the viscous force per unit volume acting on the fluid element.

The rate at which particles cross the upper surface is

$$\sim \frac{nv_T}{2}\Delta A, \qquad v_T \sim \left(\frac{kT}{m}\right)^{1/2} = \text{thermal velocity}.$$

The particles travel a distance $\sim \lambda$ before colliding and exchanging momentum with another particle, so although equal numbers of particles cross the boundary in the upward and downward direction, there is a systematic difference in momentum

$$\Delta p_x \sim 2m\lambda\frac{\partial u_x}{\partial y}$$

per particle.
The rate of change of momentum of the element due to particles crossing its boundaries is therefore

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} \sim \left[m\lambda\frac{\partial u_x}{\partial y}nv_T\Delta A\right]_{y+\Delta y} - \left[m\lambda\frac{\partial u_x}{\partial y}nv_T\Delta A\right]_y,$$

since we must subtract the momentum being taken out by particles crossing the bottom surface. Dividing by the volume $\Delta A \Delta y$ gives the force per unit volume, so

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u_x}{\partial y}\right) \sim \frac{\partial}{\partial y}\left(nm\lambda v_T\frac{\partial u_x}{\partial y}\right)$$

and using $\lambda = (n\sigma)^{-1}$ implies

$$\mu \sim \frac{mv_T}{\sigma}. \tag{36}$$

The coefficient of shear viscosity is independent of density because if the density increases then the higher flux of particles across the boundaries is countered by the shorter distance λ each particle goes, and hence the smaller amount of momentum $m\lambda\frac{\partial u_x}{\partial y}$ that it transfers.
Note also that a uniform shear ($\frac{\partial u_x}{\partial y}$ =const.) produces no net force because the drag from above is cancelled by the drag from below.

For neutral atomic hydrogen

$$\mu = 6 \times 10^{-3}\left(\frac{T}{10^4\,\text{K}}\right)^{1/2}\ g\,\text{cm}^{-1}\,\text{s}^{-1}.$$

Another frequently used quantity is the kinematic viscosity,

$$\nu \equiv \frac{\mu}{\rho} \sim v_T\lambda \sim \left(\frac{kT}{m}\right)^{1/2}\frac{1}{n\sigma} \quad [\,\text{cm}^2\,\text{s}^{-1}]. \tag{37}$$

Note from the momentum equation that $\frac{\partial u}{\partial t} \sim \nu \frac{\partial^2 u}{\partial x^2}$.

Hydrodynamics literature is also replete with references to the Reynolds number

$$\mathrm{Re} \equiv \frac{\rho u L}{\mu} = \frac{u L}{\nu} \sim \frac{u}{v_T} \frac{L}{\lambda}. \tag{38}$$

For small Re, viscous forces have an important effect in altering a flow with large velocity gradients, while for Re$\gg 1$ viscous forces can usually be ignored.
A fluid can usually remain turbulent on scales where Re is large, while viscosity damps out turbulence on scales where Re$\lesssim 1$.

4.8   Heat conduction

In typical cases, the conduction heat flux $\vec{F}$ is proportional to the temperature gradient

$$\vec{F} = -K\vec{\nabla} T,$$

where K is the coefficient of thermal conductivity. This is called Fourier's law.

For neutral gas, the coefficient is

$$K = \frac{5}{2} c_V \mu \sim \frac{k}{\sigma} \left(\frac{kT}{m} \right)^{1/2}.$$

In the case of neutral atomic hydrogen

$$K = 2 \times 10^6 \left(\frac{T}{10^4\,\mathrm{K}} \right)^{1/2} \mathrm{g\,cm\,s^{-3}\,K^{-1}}.$$

Note that the change in internal energy is proportional to $\vec{\nabla} \cdot \vec{F}$, so conduction only changes the internal energy locally when the temperature gradient is not constant.

It is useful to note that the units of K can also be written $\mathrm{erg\,s^{-1}\,cm^{-1}\,K^{-1}}$. Multiplying by a temperature gradient with units $\mathrm{K\,cm^{-1}}$ and taking a spatial derivative ($\vec{\nabla} \cdot \vec{F}$) therefore yields something with units $\mathrm{erg\,s^{-1}\,cm^{-3}}$.
Therefore $\vec{\nabla} \cdot \vec{F}/\rho$ has units of $\mathrm{erg\,s^{-1}\,g^{-1}}$, the same as $\frac{\mathrm{D}\epsilon}{\mathrm{D}t}$.