II. Relativity

The Electrodynamics of Moving Bodies

Lorentz transformations:

Translation: $t' = t$, $x' = x - A$, $y' = y$, $z' = z$
Rotation: $t' = t$, $x' = x \cos\theta + y \sin\theta$, $y' = -x \sin\theta + y \cos\theta$, $z' = z$
Lorentz boost along x-axis:

Homogeneity $\Longrightarrow$ linear
Symmetry $\Longrightarrow y' = y$, $z' = z$
$x' = a_{x'x}x + a_{x't}t$, $t' = a_{t'x}x + a_{t't}t$
$x = 0 \Longrightarrow x' = -vt' \Longrightarrow a_{x't} = -va_{t't}t$.
Consideration of a spherical light wave emitted at $t = t' = 0$ shows that
$x^2 + y^2 + z^2 = c^2t^2$ must imply $x'^2 + y'^2 + z'^2 = c^2t'^2$
if c is observer independent.
$\Longrightarrow x^2 - x'^2 + c^2(t'^2 - t^2) = 0$.
Expanding and setting coefficients to zero yields three more constraints.
Final result:
$t' = \beta(t - vx/c^2)$, $x' = \beta(x - vt)$, $y' = y$, $z' = z$, where $\beta \equiv (1 - v^2/c^2)^{-1/2}$.

Contrast to "Galilean boost": $t' = t$, $x' = x - vt$, $y' = y$, $z' = z$

Implications:

Length contraction, time dilation, velocity composition formula.

Can do electrodynamics by transformed electrostatics:

magnetic and electric fields transform, removing asymmetry between them
$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
Doppler effect, with relativistic corrections. Aberration of light.
Energy transformation. (Note that he doesn't quote $E = h\nu$.)
Radiation pressure.
Impossibility of accelerating to c.

Later: $E = mc^2$.

Space and Time (Minowski, 1908)

Great rhetoric.
Many important concepts: spacetime, worldline, past and future light cones, spacelike and timelike separations, invariant interval

spacelike: an observer could see both events as simultaneous
timelike: an observer could pass through both events
null: a light signal could pass through both events

invariant interval: $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$

The Equivalence Principle

Newtonian gravity: $\mathbf{a} = \mathbf{F}/m, \qquad \mathbf{F} = GMm\hat{\mathbf{r}}/r^2.$
Why is this unsatisfactory?

Implicitly assumes infinite speed of signal propagation.

Coincidental equality of inertial and gravitational mass.

Einstein, 1907: "The happiest thought of my life."

Influence of Gravitation . . . article, 1911, a reflection on part of 1907 review article.

Equivalence between uniform gravitational field and uniform acceleration of frame. True in mechanics. *Assume* exact equivalence.
Equivalence principle implies gravitational and inertial masses *must* be equal.
Allows extension of relativity to accelerating frames.
Implies that extension of relativity *must* involve gravity.

Third frame trick $\longrightarrow$ gravitational mass of electromagnetic energy, gravitational redshift and time dilation, bending of light (incorrect answer because ignores curvature of space)

Restatement of equivalence principle: In the coordinate system of a freely falling observer, special relativity always holds locally. No gravity.
Over larger scales, gravity can't be eliminated — tidal effects. E.g., freely falling objects in an inhomogeneous gravitational field may accelerate towards or away from each other.

Now we'll skip years of struggle . . .

MTW Summary of General Relativity

Spacetime tells matter how to move. (Along geodesic paths.)
Matter tells spacetime how to curve. (Field equation.)

Need to learn mathematical tools to describe this with precision (part of what took Einstein so long); we'll do a cursory job. My sources: Einstein, Padmanabhan, Peebles, MTW.

Tensors

Einstein: Expressions of physical laws should be independent of choice of coordinates $\Longrightarrow$generally covariant.

Tensor – a family of "linear machines," which operate on other tensors. Built of components that transform under change of coordinates in specified, linear way.

Note that coordinate transformations can be complicated. E.g., Cartesian to polar, or spatially variable "grid size," or moving with accelerated observer.

Scalar (rank-0 tensor): 1 component, invariant under transformation.
Non-relativistic examples: density, temperature, speed

Vector (rank-1 tensor): Magnitude and "direction." Components change under coordinate transformation, though vector itself does not. Dot-product with another vector yields a scalar.
Non-relativistic example: Velocity.
$\mathbf{v} \cdot \hat{\mathbf{r}} =$ speed in direction $\hat{\mathbf{r}}$.
$\mathbf{v} \cdot \mathbf{v} = \text{speed}^2$

Rank-2 tensor: Operates on vector to give a vector.
Non-relativistic example: moment of inertia tensor $\mathbf{I}$, $I_i j = \int d^3x \rho(\mathbf{x}) x_i x_j$.
If angular velocity vector is $\mathbf{o}$, angular momentum vector is
$\mathbf{L} = \mathbf{I} \cdot \mathbf{o} \quad \Longrightarrow \quad L_i = \sum_j I_{ij} o_j = I_{ij} o_j$ (Einstein summation convention)

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} o_1 \\ o_2 \\ o_3 \end{pmatrix}.$$

$\mathbf{L} \cdot \hat{\mathbf{r}} = \mathbf{I} \cdot \mathbf{o} \cdot \hat{\mathbf{r}} = I_{ij} o_j \hat{r}_i =$ angular momentum in direction $\hat{\mathbf{r}}$
In 4-d spacetime, rank-2 tensor components form a 4×4 matrix, change under coordinate transformation.

Importance of tensors to GR: Because components transform linearly under coordinate changes, if a physical law is expressed in the form **Tensor=0** (all components equal to zero), it must hold in *all* coordinate systems.

Component Transformations, Covariant and Contravariant Tensors

Consider a change of coordinates from x^μ to $x^{\overline{\nu}}(\mathbf{x})$, $\mu, \nu = 0, 1, 2, 3$. Vectors and tensors have new components in the new coordinate system.

Vector component transformations:

$$\text{Covariant}: \quad A_{\overline{\mu}} = A_\nu \frac{\partial x^\nu}{\partial x^{\overline{\mu}}} \qquad (= \sum_{\mu,\nu} A_\nu \frac{\partial x^\nu}{\partial x^{\overline{\mu}}} \text{ by summation convention})$$

$$\text{Contravariant}: \quad A^{\overline{\mu}} = A^\nu \frac{\partial x^{\overline{\mu}}}{\partial x^\nu}$$

Rank-2 tensor component transformations:

$$\text{Covariant}: \quad A_{\overline{\sigma}\overline{\tau}} = A_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\overline{\sigma}}} \frac{\partial x^\nu}{\partial x^{\overline{\tau}}}$$

$$\text{Contravariant}: \quad A^{\overline{\sigma}\overline{\tau}} = A^{\mu\nu} \frac{\partial x^{\overline{\sigma}}}{\partial x^\mu} \frac{\partial x^{\overline{\tau}}}{\partial x^\nu}$$

$$\text{Mixed}: \quad A^{\overline{\sigma}}_{\overline{\tau}} = A^\mu_\nu \frac{\partial x^{\overline{\sigma}}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\overline{\tau}}}$$

Units

For most of our discussion of GR, we will adopt the common convention of adopting units where $G = c = 1$ (see MTW, box 1.8).

4-velocity vector

A basic vector is 4-velocity $\mathbf{u}$ of an observer, the derivated of spacetime position with respect to the observer's proper time. For an observer moving with 3-velocity $\mathbf{v} = \{v^i, v^j, v^k\}$ in a Cartesian coordinate frame $\{t, x, y, z\}$,

$$u^0 = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - |\mathbf{v}|^2}}$$

$$u^j = \frac{dx^j}{d\tau} = \frac{v^j}{\sqrt{1 - |\mathbf{v}|^2}}$$

Note that $dx^j/dt = v^j$, as expected from the definition of 3-velocity.

In an observer's local Lorentz frame, the components of the 4-velocity are always $\mathbf{u} = \{1, 0, 0, 0\}$.

Metric tensor

In GR (and differential geometry), a fundamental role is played by metric tensor $g_{\mu\nu}$. (Einstein just calls it the "fundamental tensor.")

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is a scalar (invariant)
$ds^2 < 0: \ |ds| \ = \$ proper time measured by an observer passing through events
$ds^2 > 0: \ |ds| \ = \$ distance measured by an observer who sees both events as simultaneous

Equivalence principle $\Longrightarrow$in frame of freely falling observer, there is at least an infinitesimal region in which $g_{\mu\nu}$ has the special relativistic form

$$g_{\mu\nu} = \text{diag}(-1,1,1,1) \equiv \eta_{\mu\nu} \qquad \Longrightarrow \qquad ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad (c = 1 \text{ units})$$

Squared length of a vector is $|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A} = g_{\mu\nu} A^\mu A^\nu$.
Inner product is $\mathbf{A} \cdot \mathbf{B} = g_{\mu\nu} A^\mu B^\nu$.
Since 4-velocity is $\{1,0,0,0\}$ in local Lorentz frame, 4-velocity always has length=$\sqrt{-1}$.

Raising and lowering indices

Can transform contravariant to covariant indices with $g_{\mu\nu}$, e.g.
$A^\mu = g^{\mu\nu} A_\nu, \quad A_\mu = g_{\mu\nu} A^\nu, \quad A^\mu_\nu = g^{\mu\sigma} A_{\sigma\nu}.$

Covariant differentiation

Denote partial derivative by , – e.g., $\phi_{,\nu} \equiv \frac{\partial \phi}{\partial x^\nu}$.
Gradient of ϕ has components $\phi_{,\nu}$.
But $A_{\mu,\sigma}$ is *not* a tensor; doesn't transform right.
Problem is making parallel transport coordinate independent – have to "bring vectors to the same place" in a well-behaved way before taking difference.
Denote true covariant differentiation by ; instead of by , .
Scalar: $\phi_{;\alpha} = \phi_{,\alpha}$.
Vector: $A_{\mu;\beta} = A_{\mu,\beta} - \Gamma^\nu_{\mu\beta} A_\nu \equiv \frac{DA_\mu}{dx^\beta}$.
The "Christoffel symbol" or "affine connection" is

$$\Gamma^\mu_{\sigma\delta} \equiv \frac{1}{2} g^{\mu\nu} \left(-\frac{\partial g_{\sigma\delta}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\delta} + \frac{\partial g_{\nu\delta}}{\partial x^\sigma} \right) = \Gamma^\mu_{\delta\sigma}.$$

Covariant derivative of a rank-2 tensor:

$$A^\alpha_{\beta;\mu} = A^\alpha_{\beta,\mu} + \Gamma^\alpha_{\nu\mu} A^\nu_\beta - \Gamma^\nu_{\beta\mu} A^\alpha_\nu.$$

Geodesics

Suppose we have a timelike curve (tangent vector is always timelike) connecting 2 events, E_1 and E_2. Can parametrize position on curve by $s = \int_{E_1} ds$, since ds is invariant under coordinate transformations.
In a specified coordinate system, the geodesic curve, which minimizes $\int_{E_1}^{E_2} ds$, satisfies

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0, \quad \alpha = 0, 1, 2, 3.$$

Four 2nd-order differential equations that determine $x^\alpha(s)$, given initial position and 4-velocity.
Recall: 4-velocity $\mathbf{u} = \left\{ \frac{dt}{d\tau}, \frac{dx^j}{d\tau} \right\}$, in rest frame of traveling observer $= \{1, 0, 0, 0\}$.

The Riemann tensor

A rank-4 tensor built from derivatives of the metric.

Completely characterizes curvature of spacetime through the geodesic deviation equation

$$\frac{D^2 \mathbf{l}}{d\tau^2} + \mathbf{Riemann}(\mathbf{u}, \mathbf{l}, \mathbf{u}) = 0,$$

where $\mathbf{u}$ =4-velocity along geodesic, l =separation vector to nearby geodesic.
In component language

$$\frac{D^2 l^\alpha}{d\tau^2} + R^\alpha_{\beta\gamma\delta} \frac{dx^\beta}{d\tau} l^\alpha \frac{dx^\delta}{d\tau} = 0.$$

If $\mathbf{R} = 0$, spacetime is flat and geodesics do not converge or diverge. True in any coordinate system, since coordinate transformation cannot alter the fact that $\mathbf{R} = 0$.
$\mathbf{R} = 0$ is a necessary condition for there to exist coordinates in which $g_{\mu\nu}$ = constant, since $g_{\mu\nu}$ = constant $\implies \mathbf{R} = 0$.

Now we have the minimal mathematical ingredients and concepts for defining GR.

GR: Effect of geometry on matter

Suppose: special relativity holds in some finite region.

With appropriate coordinates, $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}$.
Free particles move in straight lines, at uniform velocity.
Change coordinates $\Longrightarrow g_{\mu\nu}$ change, particles follow curved paths that are independent of mass. By equivalence principle, we interpret this as motion under the influence of a gravitational field (uniform or non-uniform, depending on transformation).
Particle paths *are still geodesics*, since these are coordinate independent.

If the supposition above doesn't hold:
Retain the view that $g_{\mu\nu}$ describe the gravitational field.
Assume that freely falling particles still follow geodesics.
What else could they do? No other "special" paths.
Can derive from least-action principle and special relativity form of energy.

Implications:
Geodesic equation gives path of freely falling particle in specified coordinate system.
Geodesic deviation equation describes relative accelerations.

Example: Follow great circles on a sphere, plot separation vs. time. Looks like edge-on spacetime diagram of binary star orbit. Geodesic deviation equation is

$$\frac{d^2 l}{ds^2} + Rl = 0.$$

GR: Effect of matter on geometry

Need an equation to tell how matter produces spacetime curvature, since to get motion of particles we need the metric $g_{\mu\nu}$.
Must regain Newtonian gravity in appropriate limit $\longrightarrow$ use Poisson's equation for guidance: $\nabla^2\Phi = 4\pi G\rho$.
We want: [curvature] = [matter]

Right-hand side: the stress-energy tensor

Relativistic generalization of ρ is stress-energy tensor $\mathbf{T}$.
$-\mathbf{T}\cdot\mathbf{u}\cdot\mathbf{u} \equiv -T_{\alpha\beta}u^\alpha u^\beta =$ energy density seen by observer with 4-velocity $\mathbf{u}$.
$-\mathbf{T}\cdot\mathbf{u}\cdot\hat{\mathbf{r}} \equiv -T_{\alpha\beta}u^\alpha \hat{r}^\beta =$ component of 4-momentum density in direction $\hat{\mathbf{r}}$ in Lorentz frame defined by $\mathbf{u}$
Ideal fluid: $T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta}$.
General: all stress-energy *except* that of gravitational field.

In a Lorentz frame, equation for conservation of energy and momentum is $T^{\mu\nu}_{\ \ ,\nu} = 0$. Time derivative = spatial divergence.
E.g., perfect fluid, Newtonian limit, $|v_j| \ll 1$, $p \ll \rho$.
$T^{00} = (\rho + p)u^0 u^0 - p\eta^{00} \approx \rho \qquad (u^0 \approx 1)$.
$T^{0j} = T^{j0} = (\rho + p)u^0 u^j \approx \rho v_j$.
$T^{jk} = (\rho + p)u^j u^k + p\delta^{jk} \approx \rho v^j v^k + p\delta^{jk}$.

$$T^{00}_{\ \ ,0} + T^{0j}_{\ \ ,j} = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \qquad \text{continuity equation.}$$

Can be written in component form as

$$\frac{\partial\rho}{\partial t} + \rho\frac{\partial v^k}{\partial x^k} + v^k\frac{\partial\rho}{\partial x^k} = 0.$$

$$\begin{aligned} T^{j0}_{\ \ ,0} + T^{jk}_{\ \ ,k} &= \frac{\partial\rho v_j}{\partial t} + \frac{\partial\rho v^j v^k}{\partial x^k} + \frac{\partial p}{\partial x^j} \\ &= \rho\frac{\partial v_j}{\partial t} + v^j\left[\frac{\partial\rho}{\partial t} + v_k\frac{\partial\rho}{\partial x^k} + \rho\frac{\partial v^k}{\partial x^k}\right] + \rho v^k\frac{\partial v^j}{\partial x^k} + \frac{\partial p}{\partial x^j} = 0. \end{aligned}$$

Rewritten in vector form:

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p, \qquad \text{Euler equation.}$$

The covariant expression for energy-momentum conservation must be $T^{\mu\nu}_{\ \ ;\nu} = 0$, or $T_{\mu\nu}^{\ \ ;\nu} = 0$.

Left-hand side: the Einstein tensor

$[\quad] = \kappa T_{\mu\nu}, \quad \kappa =$ constant to be determined

$\nabla^2\Phi \sim$ *relative* accelerations $\sim$ Riemann tensor; 2nd derivatives of $g_{\mu\nu}$ are $\sim \nabla^2\Phi$, $g_{\mu\nu} \sim \Phi$,

We want

(1) A symmetric, rank-2 tensor which
(2) is built from metric and derivatives up to 2nd order ($\Longrightarrow$from $g_{\mu\nu}$ and $R_{\alpha\beta\gamma\delta}$, which is the only tensor that can be built from derivatives of the metric) and
(3) is linear in curvature (Riemann tensor) and
(4) vanishes when spacetime is flat (when $\mathbf{R} = 0$) and
(5) automatically satisfies conservation law, $[\quad]^{;\nu} = 0$.

These conditions lead uniquely to the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

where $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ is the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci (or curvature) scalar. (Notation in Einstein paper is different.)
Vanishing of $G^{;\nu}_{\mu\nu}$ is a geometrical identity known as the Bianchi identity. Conservation of energy-momentum is an automatic consequence of the way it affects spacetime.

The field equation

We have arrived at the Einstein field equation.

$$G_{\mu\nu} = 8\pi T_{\mu\nu},$$

where the constant of 8π will be justified by demanding Newtonian correspondence.
Einstein didn't know Bianchi identity, so followed a more roundabout route.
Hilbert derived from action principle based on R, the only appropriate scalar for describing curvature. Action is

$$S = \int_\Omega d^4x\sqrt{-g}\left(\mathcal{L}_{\text{matter}} - \frac{R}{16\pi G}\right).$$

Operate on both sides of the Einstein Equation (i.e., take the trace) to find

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = R - \frac{1}{2}\delta^\mu_\mu R = R - 2R = -R = 8\pi g^{\mu\nu}T_{\mu\nu} = 8\pi T.$$

The equation can therefore be written

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T),$$

which is the form (with different notation and a not yet specified constant) in equation (53) of the Einstein paper.

Solutions of the field equation

Note that $\mathbf{G} = 8\pi\mathbf{T}$ is a set of ten, second-order differential equations for the ten components of $g_{\mu\nu}$.

Second-order $\Longrightarrow$

boundary conditions matter

spacetime can be curved even where $\mathbf{T} = 0$

propagating wave solutions exist

$G_{\mu\nu}^{;\nu} = 0 \Longrightarrow$ 4 identities, so there are only six independent constraints on $g_{\mu\nu}$.

Remaining four degrees of freedom reflect freedom to choose coordinates arbitrarily.

Nonlinear $\Longrightarrow$hard to solve.

Some exact solutions, e.g.

$\mathbf{T} = 0$ everywhere $\longrightarrow$ flat spacetime, "Minkowski space"

Spherically symmetric, flat at ∞, point mass at $r = 0$ $\longrightarrow$Schwarzschild solution

Generalization to include angular momentum $\longrightarrow$Kerr solution

Homogeneous cosmologies, which we will study

In other cases, approximate.

Some relevant limits:

$g_{\mu\nu} \approx$constant, i.e. gradients can be ignored $\longrightarrow$special relativity

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ $\longrightarrow$weak field approximation

Weak field and $v \ll c$ $\longrightarrow$Newtonian limit

Correspondence to Newtonian gravity

Recall that a particle moving at constant 3-velocity $\mathbf{v}$ in a "Cartesian" frame has 4-velocity $\mathbf{u} = \{u^\alpha\}$ with

$$u^0 = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - |\mathbf{v}|^2}}, \quad u^i = \frac{dx^i}{d\tau} = \frac{v^i}{\sqrt{1 - |\mathbf{v}|^2}}.$$

In the Newtonian limit, $v \ll c$,

$$u^0 \approx 1, \quad u^i = \frac{dx^i}{d\tau} \ll 1 \quad (i = 1, 2, 3).$$

We'll need the definition of the Christoffel symbol:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}) \; = \; g_{\alpha\nu}\Gamma^\nu_{\beta\gamma},$$
$$\Gamma^\nu_{\beta\gamma} = g^{\nu\alpha}\Gamma_{\alpha\beta\gamma}.$$

Geodesic equation:

$$\begin{aligned}\frac{d^2x^i}{d\tau^2} &= -\Gamma^i_{\beta\gamma}u^\beta u^\gamma \\ &\approx -\Gamma^i_{00} \qquad (u^0 \approx 1 \gg u^i) \\ &\equiv -g^{i\alpha}\Gamma_{\alpha 00} \\ &\approx -\Gamma_{i00} \qquad (g_{\mu\nu} \approx \eta_{\mu\nu}) \\ &= -\frac{1}{2}(2h_{0i,0} - h_{00,i}) \\ &\approx \frac{1}{2}h_{00,i} \qquad (v \ll c \implies \text{small time derivatives}).\end{aligned}$$

Field equation:

Use the form

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T),$$

where $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the stress-energy tensor.
We'll adopt the stress-energy tensor of an ideal fluid,

$$\begin{aligned}T_{\alpha\beta} &= (\rho + p)u_\alpha u_\beta + pg_{\alpha\beta} \\ &\approx \mathrm{diag}(\rho, p, p, p) \qquad (u^0 \approx 1 \gg u^i,\ g_{\mu\nu} \approx \eta_{\mu\nu}).\end{aligned}$$

From the geodesic equation above, we see that we only need the 00 component in order to get particle motion.

$$\begin{aligned}R_{00} &= 8\pi(T_{00} - \frac{1}{2}g_{00}T) \\ &\approx 4\pi(2T_{00} + (-T_{00} + T_{11} + T_{22} + T_{33})) \qquad (g_{\mu\nu} \approx \eta_{\mu\nu}) \\ &= 4\pi(T_{00} + T_{11} + T_{22} + T_{33}) \\ &\approx 4\pi(\rho + 3p).\end{aligned}$$

Now we need the definition

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}.$$

With the Newtonian limit, this implies

$$\begin{aligned}R_{00} &\approx \Gamma^\alpha_{00,\alpha} - \Gamma^\alpha_{0\alpha,0} \qquad (\text{other terms are second} - \text{order}) \\ &\approx \Gamma^i_{00,i} \qquad (\text{time derivatives are small}) \\ &= g^{i\alpha}\Gamma_{\alpha 00,i} \\ &\approx \Gamma_{i00}, i \qquad (g_{\mu\nu} \approx \eta_{\mu\nu}).\end{aligned}$$

With the definition of Γ, we find

$$\begin{aligned} R_{00} &\approx \frac{1}{2}(h_{i0,0} + h_{0i,0} - h_{00,i})_{,i} \\ &\approx -\frac{1}{2}\sum_i \frac{\partial^2 h_{00}}{(\partial x^i)^2} \qquad \text{(time derivatives are small)} \\ &= -\frac{1}{2}\nabla^2 h_{00}. \end{aligned}$$

If we identify $h_{00} = -2\Phi$, we find

$$\begin{aligned} \frac{d^2 x^i}{d\tau^2} &= \frac{1}{2}h_{00,i} = -\frac{\partial \Phi}{\partial x^i}, \\ \nabla^2 \Phi &= -\frac{1}{2}\nabla^2 h_{00} = 4\pi(\rho + 3p). \end{aligned}$$

For a non-relativistic fluid, $p \ll \rho$, and we get the equation of motion of a particle moving under the influence of a gravitational potential Φ that obeys Poisson's equation.

Tests of GR

- yields Newtonian gravity in appropriate limit
- precision tests of equivalence principle
- precession of Mercury – the key from Einstein's point of view
- bending of light – historically important
- gravitational redshift
- higher-order solar system tests $\Longrightarrow$measured values of "post-Newtonian parameters" agree with GR predictions
- binary pulsars:

  gravity wave dissipation rate – very strong test
  precession of orbit in an external system
  gravitational time delay, effects up to $\sim (v/c)^3$

Other low precision tests: structure of dense stars, gravitational lensing

Fails badly in description of outer parts of galaxies and in galaxy groups and clusters. We'll interpret this failure as evidence for dark matter, but we should remember that the right interpretation could be different.

(Is the historical precedent Mercury, Neptune, or Pluto?)

Despite the excellent confirmation of GR (with the dark matter caveat), application to cosmology requires a gigantic extrapolation, can't rest on empirical basis of small-scale tests. [One could make a similar comment about the strong field regime – black holes, etc.]

The Einstein cosmological model

Cosmological Considerations on the General Theory of Relativity (Einstein, 1917)

Einstein recognizes the extrapolation point made above and *modifies* the field equation:

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \longrightarrow R_{\mu\nu} - \lambda g_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T).$$

In terms of our discussion, he drops the requirement that the left-hand side of the field equation vanish when space is flat (when $R_{\alpha\beta\gamma\delta} = 0$).

Why?

Field equation must be supplemented by boundary conditions at ∞.

Finite mass distribution with $g_{\mu\nu} = \eta_{\mu\nu}$ at ∞ has 2 problems:

(1) System evaporates, like a globular cluster

(2) Inertia is not determined by mass distribution alone (violates Mach's principle)

Could avoid by having an infinite potential barrier, but this violates "the most important fact that we draw from experience," that relative velocities of stars are $\ll c$.

Alternative: finite, bounded universe like surface of sphere (large scale approximation).

Assume homogeneous: "we are prompted to the hypothesis that ρ is to be independent of locality."

Derives metric from homogeneity condition and assumption that universe is static.

Doesn't fit field equation. Adds cosmological term.

λ must have specified value so that universe is in equilibrium.

It was later pointed out that this equilibrium is unstable.

De Sitter, Einstein, Friedmann, Lemaître, and others later went on to discuss expanding cosmologies.

Einstein abandoned the cosmological term for good when the cosmic expansion was discovered (in 1929). He is reputed to have called it "the greatest blunder of my life."

The cosmological constant idea has never completely gone away. It has been especially prominent in the last 20 years, but it is now viewed as part of $T_{\mu\nu}$.

Instead of

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \qquad G_{\mu\nu} = 8\pi(T_{\mu\nu}^{\text{matter}} + T_{\mu\nu}^{\text{VAC}}), \quad \text{where}$$
$$T_{\mu\nu}^{\text{VAC}} \equiv \Lambda g_{\mu\nu}/8\pi.$$

$T_{\mu\nu}^{\text{VAC}}$ is the stress-energy tensor of a "false vacuum" or "scalar field" with equation of state $p = -\rho$. We'll reencounter this idea when we get to inflation.