II. General Relativity

Suggested Readings on this Section (All Optional)

For a quick mathematical introduction to GR, try Chapter 1 of Peacock.

For a brilliant historical treatment of relativity (special and general) and Einstein's contributions to physics: *Subtle is the Lord*, by Abraham Pais, Oxford University Press.

For Einstein's original papers on relativity, see *The Principle of Relativity*, A. Einstein et al., published by Dover Books. The 1905 paper *On the Electrodynamics of Moving Bodies* is perhaps the best scientific paper ever written. The three papers that follow this in the book are well worth reading. So is the 1916 paper on General Relativity, but it is more challenging (and best read in concert with the Pais book).

Special Relativity

Postulates of theory:

1. There is no state of "absolute rest."
2. The speed of light in vacuum is constant, independent of state of motion of emitter.

(Point 2 could really be subsumed into point 1.)

Implies: Simultaneity of events and spatial separation of events depend on state of motion of observer.

Relation between coordinate systems x, y, z, t and x', y', z', t' of uniformly moving observers described by Lorentz transformations.

For a reference frame moving at constant velocity v in $+x$ direction, the *Galilean* transformation is

$$\begin{aligned} t' &= t \\ x' &= (x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

The *Lorentz* transformation is

$$\begin{aligned} t' &= \gamma \left(t - \frac{vx}{c^2} \right) \\ x' &= \gamma (x - vt) \\ y' &= y \\ z' &= z, \end{aligned}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

Derived by considering a spherical light wave emitted at $t = t' = 0$, for which
$x^2 + y^2 + z^2 = c^2 t^2$ must imply $x'^2 + y'^2 + z'^2 = c^2 t'^2$
if c is observer independent.

Observers in relative motion disagree on the spatial separation $\Delta l = \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]^{1/2}$ and the time separation Δt between the same pair of events.

But they agree on the "spacetime interval" $(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ between events.

Analogy: Stationary observers with rotated reference frames disagree on Δx, Δy but agree on $\Delta l = \left[(\Delta x)^2 + (\Delta y)^2\right]^{1/2}$.

The Equivalence Principle

"Special" relativity restricted to uniformly moving observers. Can it be generalized?

Newtonian gravity: $\mathbf{a} = \mathbf{F}/m$, $\mathbf{F} = -GMm\hat{\mathbf{r}}/r^2$.

Why is this unsatisfactory?

Implicitly assumes infinite speed of signal propagation.

Coincidental equality of inertial and gravitational mass.

Einstein, 1907: "The happiest thought of my life." If I fall off my roof, I feel no gravity.

Equivalence between uniform gravitational field and uniform acceleration of frame. True in mechanics. *Assume* exact equivalence, i.e., for electrodynamics as well.

Equivalence principle implies gravitational and inertial masses *must* be equal.

Allows extension of relativity to accelerating frames.

Implies that extension of relativity *must* involve gravity.

Third frame trick $\longrightarrow$ gravitational mass of electromagnetic energy, gravitational redshift and time dilation, bending of light (incorrect answer because ignores curvature of space)

Restatement of equivalence principle: In the coordinate system of a freely falling observer, special relativity always holds locally (to first order in separation). No gravity.

Over larger scales (second order in separation), gravity doesn't vanish in a freely falling frame — tidal effects. E.g., freely falling objects in an inhomogeneous gravitational field may accelerate towards or away from each other.

General Relativity

With aid of equivalence principle, can change relativity postulate from "There is no absolute rest frame" to "There is no absolute set of inertial frames."

Uniform acceleration can be treated as uniform gravitational field.

Freely falling particles follow geodesic paths in curved spacetime.

A geodesic path is a path of shortest distance, or more generally of shortest spacetime interval. In flat space, a straight line; in flat spacetime, a straight line at constant velocity.

Distribution of matter (more generally, stress-energy) determines spacetime curvature.

Misner, Thorne, and Wheeler's catchy summary of GR:

Spacetime tells matter how to move. (Along geodesic paths.)
Matter tells spacetime how to curve. (Field equation.)

Compare to equivalent description of Newtonian gravity:
Gravitational force tells matter how to accelerate. ($F = ma$, or $\mathbf{g} = \vec{\nabla}\Phi$.)
Matter tells gravity how to exert force. ($F = GMm/r^2$, or $\nabla^2\Phi = 4\pi G\rho$.)

Warning: remainder of relativity section will mostly use units $G = c = 1$.

Metric tensor

In GR (and differential geometry), a fundamental role is played by metric tensor $g_{\mu\nu}$.
Squared length of a vector is $|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A} = g_{\mu\nu}A^\mu A^\nu$ (note Einstein summation convention).
Inner product is $\mathbf{A}\cdot\mathbf{B} = g_{\mu\nu}A^\mu B^\nu$.
Metric tensor is symmetric, so has 10 independent components rather than 16.
Note that $g_{\mu\nu}$ is, in general, a function of spacetime position.

Spacetime interval between two events separated by small dx^μ ($\mu = 0, 1, 2, 3$) is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu.$$

In special relativity, $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

ds^2 is a scalar, hence independent of coordinate system.
$ds^2 < 0: \ |ds| \ =$ proper time measured by an observer passing through events
$ds^2 > 0: \ |ds| \ =$ distance measured by an observer who sees both events as simultaneous
Can integrate $s = \int ds$ to get interval along a path between widely separated events.

Observers with different coordinate systems (e.g., moving relative to each other in arbitrary ways) disagree on values of dx^μ, $g_{\mu\nu}$.
All agree on value of ds^2.

GR: Effect of geometry on matter

Einstein's argument:

Equivalence principle $\Longrightarrow$in frame of freely falling observer, there is at least an infinitesimal region in which $g_{\mu\nu}$ has the special relativistic form

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \equiv \eta_{\mu\nu} \qquad \Longrightarrow \qquad ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad (c = 1 \text{ units})$$

Here "infinitesimal" really means "to first order in the separation dx^μ from the observer"; in a non-uniform gravitational field, $g_{\mu\nu}$ departs from $\eta_{\mu\nu}$ at second order in dx^μ.

Suppose that special relativity holds in some finite region.
With appropriate coordinates, $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}$.

Free particles move in straight lines, at uniform velocity, i.e., along *geodesics*.

Change coordinates $\Longrightarrow g_{\mu\nu}$ change, particles follow curved paths that are independent of mass. By equivalence principle, we interpret this as motion under the influence of a gravitational field (uniform or non-uniform, depending on transformation).

Particle paths *are still geodesics*, since these are coordinate independent.

If the supposition above doesn't hold:

Retain the view that $g_{\mu\nu}$ describe the gravitational field.

Assume that freely falling particles still follow geodesics.

What else could they do? No other "special" paths.

Can derive from least-action principle and special relativity form of energy.

Bottom line: the equation of motion for freely falling particles in a specified coordinate system is just the equation for geodesic paths. In practice, this equation represents four 2nd-order differential equations that determine $x^\alpha(s)$, given initial position and 4-velocity:

$$\frac{d^2x^\alpha}{d\tau^2} + F\,[\text{metric}]\,\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0.$$

The geodesic equation is the relativistic analog of the Newtonian equation $\mathbf{g} = -\vec{\nabla}\Phi$.

The metric $g_{\mu\nu}$ is the relativistic generalization of the gravitational potential.

In units with $c = 1$, the departure of $g_{\mu\nu}$ from the Minkowski form $\eta_{\mu\nu}$ is $\sim v^2/c^2$, where v is the characteristic speed in this gravitational potential.

GR: Effect of matter on geometry

Need an equation to tell how matter produces spacetime curvature, since to get motion of particles we need the metric $g_{\mu\nu}$.

Must regain Newtonian gravity in appropriate limit $\longrightarrow$ use Poisson's equation for guidance: $\nabla^2\Phi = 4\pi G\rho$.

We want: [curvature] = [mass-energy density]

Various lines of argument lead to the **Einstein field equation**

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$

The quantity $T_{\mu\nu}$ is the *stress-energy tensor*, the relativistic generalization of ρ.

$-\mathbf{T}\cdot\mathbf{u}\cdot\mathbf{u} \equiv -T_{\mu\nu}u^\mu u^\nu$ = energy density seen by observer with 4-velocity $\mathbf{u}$.

$-\mathbf{T}\cdot\mathbf{u}\cdot\hat{\mathbf{r}} \equiv -T_{\mu\nu}u^\mu \hat{r}^\nu$ = component of 4-momentum density in direction $\hat{\mathbf{r}}$ in Lorentz frame defined by $\mathbf{u}$

For an ideal fluid, $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$.

The quantity $G_{\mu\nu}$ is the *Einstein tensor*, a symmetric, rank-2 tensor that is built from the metric and its derivatives up to 2nd order.

$\rho \sim \nabla^2\Phi \sim$ *relative* accelerations $\sim$ 2nd derivatives of $g_{\mu\nu}$
The Einstein tensor is linear in the curvature and vanishes when spacetime is flat.

The constant of 8π is determined by demanding correspondence to Newtonian gravity in the appropriate limit.

Energy-Momentum Conservation

In a Lorentz frame, equation for conservation of energy and momentum is $\sum \frac{d}{dx^\nu} T^{\mu\nu} = 0$. Time derivative = spatial divergence.

Covariant generalization of this is the general expression for energy-momentum conservation.

Vanishing of $\sum \frac{d}{dx^\nu} G_{\mu\nu} = 0$ is a geometrical identity known as the Bianchi identity.

In GR, conservation of energy-momentum is an automatic consequence of the Einstein equation: energy-momentum must be conserved because of the way it affects spacetime.

Solutions of the field equation

Note that $G_{\mu\nu} = 8\pi T_{\mu\nu}$ is a set of ten, second-order differential equations for the ten components of $g_{\mu\nu}$.

Second-order $\Longrightarrow$

boundary conditions matter

spacetime can be curved even where $T_{\mu\nu} = 0$

propagating wave solutions exist

Four geometrical identities (the Bianchi identities) constrain $G_{\mu\nu}$, so there are only six independent constraints on $g_{\mu\nu}$.

Remaining four degrees of freedom reflect freedom to choose coordinates arbitrarily.

Freedom to choose coordinates $\Longrightarrow$same spacetime can "look" very different physically for different choices of observers. The Milne cosmology of problem set 1 will be an example of this.

Nonlinear $\Longrightarrow$hard to solve.

Some exact solutions, e.g.

$\mathbf{T} = 0$ everywhere $\longrightarrow$ flat spacetime, "Minkowski space"

Spherically symmetric, flat at ∞, point mass at $r = 0 \longrightarrow$ Schwarzschild solution

Generalization to include angular momentum $\longrightarrow$ Kerr solution

Homogeneous cosmologies, which we will study

In other cases, approximate.

Some relevant limits:

$g_{\mu\nu} \approx$constant, i.e. gradients can be ignored $\longrightarrow$ special relativity
$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1 \longrightarrow$ weak field approximation
Weak field and $v \ll c \longrightarrow$ Newtonian limit

The Newtonian Limit

A first-order perturbative calculation in this Newtonian limit yields

$$\frac{d^2x^i}{d\tau^2} \approx \frac{1}{2}\frac{d}{dx^i}h_{00},$$

for the equation of motion.

The corresponding Newtonian equation of motion is

$$\frac{d^2x^i}{d\tau^2} = -\frac{\partial \Phi}{\partial x^i},$$

($\mathbf{g} = -\vec{\nabla}\Phi$), so we should identify $h_{00} = -2\Phi$.

In the Newtonian limit the ideal fluid stress-energy tensor is $T_{\mu\nu} \approx \mathrm{diag}(\rho, p, p, p)$, and the field equation then yields the result

$$-\frac{1}{2}\nabla^2 h_{00} = 4\pi(\rho + 3p),$$

and thus

$$\nabla^2\Phi = -\frac{1}{2}\nabla^2 h_{00} = 4\pi(\rho + 3p).$$

For a non-relativistic fluid, $p \ll \rho$, and we get the equation of motion of a particle moving under the influence of a gravitational potential Φ that obeys Poisson's equation.
The $3p$ contribution implies that radiation (with $p = \rho/3$) has a stronger gravitational pull than matter (for the same energy density), and that a fluid with $p < -\rho/3$ can exert gravitational push.

Tests of GR

- yields Newtonian gravity in appropriate limit
- precision tests of equivalence principle
- precession of Mercury – the key from Einstein's point of view
- bending of light – historically important
- gravitational redshift
- higher-order solar system tests $\Longrightarrow$measured values of "post-Newtonian parameters" agree with GR predictions
- binary pulsars:

gravity wave dissipation rate – very strong test
precession of orbit in an external system
gravitational time delay, effects up to $\sim (v/c)^3$

Other low precision tests: structure of dense stars, gravitational lensing

Despite these impressive tests, application to cosmology requires gigantic extrapolation in length and time scale.

Can't rest comfortably on empirical basis of small-scale tests.

Cosmological models based on GR are impressively successful, but they require two strange ingredients: dark matter and dark energy.

Existence of these ingredients could be an indication that GR is breaking down in some way on cosmological scales, though we will generally take the view that it is not.