Problem Set 5: Inflation
Due Wednesday, November 21

For parts 1, 2, and 4, we will work at an order-of-magnitude level, so you don't need to keep track of factors of 2, π, etc. However, we'll keep track of these factors in part 3.

Part 1: The Horizon Problem

For a start, assume a standard FRW universe, which at early times is radiation dominated. Using your result from Problem Sets 3/4 that

$$t(T) \approx 1 \left(\frac{kT}{\mathrm{MeV}} \right)^{-2} \mathrm{s}, \tag{1}$$

what is the approximate size $d_{\mathrm{hor}}(t_{\mathrm{GUT}})$ of the particle horizon at the time t_{GUT} when the temperature falls to $kT = kT_{\mathrm{GUT}} \approx 10^{15}\,\mathrm{GeV}$, the temperature at which the grand unified phase transition is expected to occur? Express your answer in light-seconds.

The CMB temperature today is $kT \sim 10^{-4}\,\mathrm{eV}$. Assuming standard FRW cosmology, what is the physical size of $d_{\mathrm{hor}}(t_{\mathrm{GUT}})$ today, accounting for the expansion from $t = t_{\mathrm{GUT}}$ to t_0?

What is the (approximate) physical distance to the CMB last scattering surface, in light-seconds? (Remember that eq. [1] only applies to a radiation-dominated universe.) What is the ratio of this distance to the present day value of $d_{\mathrm{hor}}(t_{\mathrm{GUT}})$?

Now assume that inflation occurs when $T \approx T_{\mathrm{GUT}}$. During inflation the universe expands as $a(t) \propto e^{Ht}$, and the temperature remains at $T = T_{\mathrm{GUT}}$ instead of dropping as a^{-1}. (More accurately, the radiation temperature drops during inflation, but it is restored to $T \approx T_{\mathrm{GUT}}$ at the end of inflation when the inflaton field decays and converts its energy into photons and other relativistic particles.)

How many e-folds of inflation are required to solve the horizon problem, i.e., to ensure that the entire region out to the last scattering surface was in one causally connected patch before inflation began?

From now on, we will refer to this minimum number of e-folds as N_{min}.

Part 2: The Flatness Problem

Suppose that before inflation begins the gravitational and curvature terms in the Friedmann equation are of similar order, and therefore

$$H^2 \sim \frac{8\pi G}{3}\rho \sim \frac{c^2}{a^2}. \tag{2}$$

What is the approximate value of the curvature radius a at time t_{GUT}, just before inflation begins?

If the number of e-folds is the minimum number N_{min} required to solve the horizon problem, what is the curvature radius today?

Is it possible for inflation to solve the horizon problem and still produce a universe with Ω measurably different from one today? Is it likely for this to happen? (Here Ω represents the contribution of all energy components — matter, radiation, dark energy — but it does not include Ω_k, since it would then equal one by definition.)

After reheating at the end of inflation, the entropy within the curvature radius is

$$S \sim \left(\frac{kT_{\rm GUT}}{\hbar c}\right)^3 a^3. \tag{3}$$

What is the value of S assuming the minimum number of e-folds? (Remember that $\hbar c = 1.97 \times 10^{-5}\mathrm{eV} - \mathrm{cm}$.)

Part 3: Constraints on the Initial Conditions

Assume that during inflation, the energy density is dominated by the potential energy $V(\phi)$ of a scalar field ϕ. As in Problem Set 4, we'll adopt high energy physics units with $\hbar = c = 1$ and $G = m_{\rm Pl}^{-2}$, where $m_{\rm Pl} = (\hbar c/G)^{1/2}$ is the Planck mass. The Friedmann equation during the inflation era is therefore

$$H^2 = \frac{8\pi}{3m_{\rm Pl}^2} V(\phi). \tag{4}$$

Note that H, $m_{\rm Pl}$, and ϕ have units of GeV and that $V(\phi)$ has units of GeV^4.

We'll assume that ϕ obeys the slow roll condition discussed in class, which is necessary for inflation to occur. The evolution equation is therefore

$$3H\dot{\phi} = -V'(\phi), \tag{5}$$

where $V'(\phi) = dV/d\phi$.

Assume that ϕ starts at time t_i at some initial value ϕ_i and rolls until $\phi = 0$, $V(\phi) = 0$, at which point inflation ends. Argue that the number of e-folds of inflation during this evolution is

$$N = \int_{t_i}^{t_{\rm end}} H(t)dt = \int_{\phi_i}^{0} H(\phi)dt = \frac{8\pi}{m_{\rm Pl}^2}\int_0^{\phi_i} \frac{V(\phi)}{V'(\phi)}d\phi. \tag{6}$$

(Hint: use the relation between dt, $\dot{\phi}$, and $d\phi$.)

Now assume a specific form for the potential: $V(\phi) = \lambda\phi^4$, where λ is a dimensionless "coupling constant." Show that the necessary condition for obtaining $N > N_{\rm min}$ e-folds of inflation is

$$\phi_i > \left(\frac{N_{\rm min}}{\pi}\right)^{1/2} m_{\rm Pl}. \tag{7}$$

For the value of $N_{\rm min}$ obtained in Part 1, what is the minimum value of ϕ_i required to get enough inflation to solve the horizon problem?

Part 4: Primordial Density Fluctuations

For the $V(\phi) = \lambda\phi^4$ potential, show that the number of e-folds is

$$N \sim \frac{H^2}{\lambda\phi^2}, \tag{8}$$

where we have used the approximation that the inflationary epoch is characterized by a single value of $\phi \sim \phi_i$ and $H \sim H(\phi_i)$.

Now assume that the rolling scalar field experiences quantum fluctuations of typical magnitude $\delta\phi$. These fluctuations cause different regions of the universe to expand for slightly different amounts of time during inflation, with fluctuations $\delta t = \delta\phi/\dot{\phi}$. Since $a \propto e^{Ht}$, these δt fluctuations lead to volume fluctuations $\delta V \sim (\delta a)^3 \sim e^{3H\delta t} \sim 3H\delta t$, where we have assumed $H\delta t \ll 1$. The typical density fluctuations present on the horizon scale at the end of inflation are therefore

$$\delta_H \sim H\delta t. \tag{9}$$

(This argument is obviously *very* rough, and more detailed versions appear in the textbooks, e.g., Peacock or Peebles. Formidably proper treatments can be found in the literature.)

Since H is the only energy scale in the problem, and we are using units with $\hbar = 1$, we can reasonably guess that $\delta\phi \sim H$, and a proper quantum field theory calculation shows this to be true.

Show that

$$H\delta t \sim \frac{H^3}{V'}. \tag{10}$$

Use this result, the result (8) for N, the minimum number of e-folds from Part 1, and the fact that CMB measurements show $\delta_H \sim 10^{-5}$ to derive an upper limit on the dimensionless coupling constant λ. (Hint: first show that $H\delta t \sim \lambda^{1/2}N^{3/2}$.)