Problem Set 5: Spherical Collapse in an $\Omega = 1$ Universe

Due *Friday* April 5

The evolution of a spherically symmetric, overdense perturbation in an $\Omega_m = 1$ universe can be solved analytically up to the point of collapse, which makes it a very useful model despite its geometrical idealization. The evolution can be viewed from a Newtonian point of view, but the relativistic derivation is more broadly applicable. The basic trick is to recognize that, as a consequence of Birkhoff's theorem (in a spherically symmetric universe, only the interior mass matters), the perturbation itself must follow the equations of a $k = +1$ Friedmann universe.

The spherical collapse model is discussed in a famous paper by Gunn & Gott (1972, Astrophysical Journal, vol. 176, p. 1) and some cosmology textbooks. If you want a reference that may be helpful in working through this problem set, try sections 2 and 3 of Shapiro, Iliev, & Raga (1999, MNRAS 307, 203).

1. Perturbation overdensity

Consider an $\Omega_m = 1$ universe containing a single, spherically symmetric, homogeneous overdense region (the perturbation). The background universe is described by the Friedmann equation for $k = 0$ and the perturbation by the Friedmann equation for $k = +1$ (since its density exceeds the critical density). The equations for the evolution of the background universe $a(t)$ and for the radius of the perturbed region $r(t_p)$ can therefore be written in the parametric form (see, e.g., course notes, §IV, or §6.1 of Ryden):

$$a = \frac{1}{2} a_* \eta^2, \qquad t = \frac{1}{6}\frac{a_*}{c}\eta^3$$
$$r = r_*(1 - \cos\eta_p), \qquad t_p = \frac{r_*}{c}(\eta_p - \sin\eta_p).$$

Use the conditions that (a) t and t_p must be equal, and (b) the perturbation vanishes at high redshift ($\rho_p \to \rho$ as η and $\eta_p \to 0$) to derive an expression for the overdensity ρ_p/ρ as a function of η_p. You may find it useful to introduce the quantity q which is the radius the perturbed region would have had if it had expanded at the unperturbed rate (i.e., if had not been overdense).

Hint: Use Taylor expansions.
Big hint: The answer you are trying to derive is:

$$\frac{\rho_P}{\rho} = \left(\frac{q}{r}\right)^3 = \frac{9}{2}\left(\eta_P - \sin\eta_P\right)^2\left(1 - \cos\eta_P\right)^{-3} \ . \qquad (1)$$

If you can't derive equation (1), assume it to be correct and go on to later parts of the problem.

2. The linear regime

Show that the density contrast

$$\frac{\delta\rho}{\rho} \equiv \frac{\rho_P - \rho}{\rho} = \frac{3}{20}\eta_P^2 \propto t^{2/3} \qquad (2)$$

when $\eta_p \ll 1$.

Show that the dimensionless velocity perturbation for $\eta_p \ll 1$ is

$$\delta_V \equiv \frac{v - Hr}{Hr} = -\frac{1}{3}\left(\frac{\delta\rho}{\rho}\right), \qquad (3)$$

where $v = \dot{r}$ is the perturbation's expansion velocity and H is the Hubble parameter of the background universe.

Hint: You need to keep doing Taylor expansions in η_P until the quantities you are trying to compute don't vanish. It will be worth it in the end.

3. Turnaround

Let δ_i denote the density contrast $\delta\rho/\rho$ at some early time t_i ($\eta_p \ll 1$), and let r_i denote the perturbation radius at this time.

Show that the perturbation expands to a maximum or "turnaround" radius

$$r_{ta} = r_i \left(\frac{5}{3}\delta_i\right)^{-1} \quad \text{at time} \quad t_{ta} = t_i \left(\frac{3\pi}{4}\right)\left(\frac{5}{3}\delta_i\right)^{-3/2} . \tag{4}$$

Show that the overdensity at turnaround is $\rho_p/\rho \approx 5.5$.

[Extra credit: The expressions for turnaround radius and time differ from those of Gunn & Gott (1972) by the factors of 5/3. Why?]

4. Collapse

Show that the perturbation collapses to zero radius at $t_c = 2t_{ta}$.

Going back to your answer in Part 2, show that *if* you had used linear perturbation theory to compute the amplitude of the perturbation at time t_c instead of this non-linear spherical collapse model, you would have concluded that

$$\delta_{\text{lin}}(t_c) \approx 1.69 . \tag{5}$$

This is a famous and useful number, often referred to as the collapse threshold and denoted δ_c.

Hint: What is the relation between δ and η_P in the linear regime? What is the value of η_P at collapse?

5. Virialization

If the perturbation were truly spherically symmetric, it could collapse to a black hole, but even the smallest departures from spherical symmetry will prevent this. Instead, it is usually assumed that the perturbation collapses to form a spherical system in approximate virial equilibrium at $t = t_c$.

Assume that non-radial velocities are negligible at turnaround, so at $t = t_{ta}$ the perturbation has zero kinetic energy.

Argue that after the perturbation collapses and reaches virial equilibrium (potential energy $= -2 \times$ kinetic energy) it will have a radius $r_c \approx r_{ta}/2$.

If virialization occurs instantaneously at $t = t_c$, what will the density of the virialized object be relative to the mean density of the universe at $t = t_c$?

This is another famous and useful number.

[For the purposes of this problem, assume that the virialized object that forms after collapse is homogeneous (and spherical). Comment on the importance/believability of this assumption if you wish.]

6. Isothermal halo

A more realistic (though still idealized) model is that the post-collapse dark matter halo is a singular isothermal sphere, with $\rho(r) \propto r^{-2}$ and enclosed mass $M(< r) \propto r$. Show that in this case the *half-mass radius* of the virialized halo is

$$r_{1/2} = \frac{5}{12} r_{ta} \ .$$

7. Applications

If all went well, then in Part 5 you derived the well-known lore that a spherical perturbation in an $\Omega_m = 1$ universe has a mean overdensity $M/(4\pi R^3/3) = 178\rho_{\rm crit}$ after it collapses and virializes, where R is the "virial radius." Since the details of the physical argument are rough, and there is some cosmology dependence in any case, it is common to refer to the virial radius as R_{200} within which the mass is M_{200} and the mean overdensity is $200\rho_{\rm crit}$.

With this definition, show that the virial radius of a dark matter halo of circular velocity $V_{200} = (GM/R_{200})^{1/2}$ is

$$V_{200} = 10 H_0 R_{200} = 1000h \left(\frac{R_{200}}{1\,\rm Mpc} \right) \,\rm km\,s^{-1} \tag{6}$$

and that

$$M_{200} = \frac{V_{200}^3}{10 G H_0} = 2.3 \times 10^{14} h^{-1} M_\odot \left(\frac{V_{200}}{1000\,\rm km\,s^{-1}} \right)^3 \ , \tag{7}$$

where $h \equiv H_0/100\,\rm km\,s^{-1}\,Mpc^{-1}$.

What are the virial radius and virial mass of the Milky Way's dark matter halo?

Hint: In a useful set of units

$$G = 4.30 \times 10^{-9} (\,\rm km\,s^{-1})^2\,Mpc\,M_\odot^{-1} \ . \tag{8}$$

Note: In the cosmological literature, halo boundaries and masses are sometimes described in terms of R_{200c}, the radius within which the density is 200 times $\rho_{\rm crit}$, and sometimes in terms of R_{200m}, the radius within which the density is 200 times the mean matter density $\bar{\rho}_m$. Since we live in a universe with $\Omega_m \neq 1$, the two definitions are different. Each has something to recommend it, but it can be confusing to keep track of what is being used in a given paper.