Astronomy 294 – Life in the Universe
Autumn Quarter 2008 – Prof. Gaudi
Homework #2 Solutions

Astronomers surveying the outer solar system discover a new planet that they name Rocky. Astronomers estimate that Rocky has an orbital period around the Sun of 64 years and is on a circular orbit.

Question 1 (20 points)

What is Rocky's semi-major axis?

a) 64 AU
b) 16 AU
c) 4,096 AU
d) 512 AU
e) 8 AU

Kepler's third law states that,

$$[P(\text{years})]^2 = [a(\text{AU})]^3 \,,$$

where P is the period and a is the semi-major axis. We can solve for a to find,

$$a(\text{AU}) = [P(\text{years})]^{2/3} \,.$$

We can now insert P=1000 years, and find,

$$a = [P(\text{years})]^{2/3}\,\text{AU} = (64)^{2/3}\,\text{AU} = 16\,\text{AU} \,.$$

Answer b) 16 AU

Question 2 (20 points)

What is the apparent brightness of the Sun on Rocky relative to the apparent brightness on the Earth?

a) 16 times brighter
b) 262,144 times fainter
c) 256 times fainter
d) 64 times fainter
e) 4096 times fainter

This is essentially the same problem as Question 3 of Homework #1. As one goes further away from the Sun, its luminosity is spread out over a larger and larger surface area. We can picture an imaginary sphere of radius d centered on the Sun, upon which the Sun's light is incident. The surface area of this sphere is $4\pi d^2$. The brightness B is the amount of light from the Sun that hits this sphere in a given unit area,

$$B = \frac{L}{4\pi d^2},$$

so the ratio of the brightness at the Earth to that at the distance of Fred is just the inverse square of the distances:

$$\frac{B_{George}}{B_{Earth}} = \left(\frac{d_{Earth}}{d_{Marvin}}\right)^2$$

So the ratio of brightness is $(16)^{-2}=1/(256)$.

Answer: c) 256 times fainter

Question 3 (20 points)

Assuming that Rocky has the same average albedo as the Earth, what is the equilibrium temperature of Rocky relative to the equilibrium temperature of the Earth?

a) Rocky is 16 times warmer
b) Rocky is 4096 times cooler
c) Rocky is 64 times cooler
d) Rocky is 4 times cooler
e) Rocky is the same temperature

The equilibrium temperature is given by,

$$T = \left(\frac{L}{16\pi\sigma}\right)^{1/4} d^{-1/2}.$$

This does not include the effect of the albedo (amount of light reflected by the planet), but since we are assuming this is the same as the Earth, we can ignore this for now. Note that the first part of the equation for the equilibrium temperature, namely $\left(\frac{L}{16\pi\sigma}\right)^{1/4}$, is the same for both the Earth and Rocky. Therefore, all we have to worry about is the ratio of distances. Therefore we have,

$$\frac{T_{Earth}}{T_{George}} = \left(\frac{d_{George}}{d_{Earth}}\right)^{1/2} = \left(\frac{16\text{AU}}{1\text{AU}}\right)^{1/2} = 4.$$

Thus Earth is 4 times hotter than Rocky.

Answer d) Rocky is 4 times cooler

Question 4 (20 points)

Now assume that Rocky is a perfect absorber (zero albedo) and absorbs the Sun's energy on one hemisphere but emits energy over his entire surface. What is Rocky's equilibrium temperature? (*Hint: You can find the relevant formula in my lecture notes on the class webpage).*

a) 69.5 K
b) 17.4 K
c) 1.1 K
d) 4.3 K
e) 34.8 K

In class I derived the formula:

$$T \sim 278K\left(\frac{a}{\mathrm{AU}}\right)^{-1/2}.$$

This assumes a perfect absorber and that the energy is absorbed on one hemisphere but emitted over the entire surface, and also assumes that the distance from the Sun (d) is equal to its average distance (the semi-major axis a). This is appropriate for Rocky, because it is on a circular orbit, and so its distance is always equal to its semi-major axis. In Question 1 we computed that Rocky's semi-major axis is 16 AU. So, Rocky's equilibrium temperature is then,

$$T \sim 278K\left(\frac{a}{\mathrm{AU}}\right)^{-1/2} = 278K\left(\frac{16\mathrm{AU}}{\mathrm{AU}}\right)^{-1/2} = 278K/4 = 69.5K.$$

Answer a) 69.5 K

Question 5 (10 points)

Scientists discover that Rocky's surface is actually covered with snow (water ice). Given what you've calculated for the equilibrium temperature, is this a surprising result?

a) Yes
b) No

We discussed in class the freezing point of water for various pressures. For pressures of one atmosphere, this is around 273K or 0^{o} C. This temperature varies somewhat with pressure, but never gets anywhere near Rocky's temperature of ~70K. Furthermore, we discussed that the outer solar system was enriched in ices during its formation. Therefore, it's not surprising that Rocky is covered with water ice.

Answer b) No

Question 6 (10 points)

If Rocky is indeed covered with snow, is his equilibrium temperature higher or lower than what you calculated in Question 4?

a) Higher
b) Lower
c) Rocky's equilibrium temperature is the same.

Snow reflects a lot of light. Therefore, its albedo is quite high. As we discussed in class and as given in my notes, the formula for the equilibrium temperature that we used in Question 4 ignores albedo and assumes the planet is a perfect absorber. Higher albedo means lower temperature (at fixed distance), because the planet is not absorbing as much energy. Therefore, Rocky's temperature must be lower than calculated in Question 4.

Answer b) Lower

Extra Credit Questions:

In class I said that the velocity needed to escape the gravitational pull of a planet of mass M and radius R is,

$$v_E = \sqrt{\frac{2GM}{R}}$$

Where G is the gravitational constant.

Question 7 (10 points)

Assuming that all planets have the same uniform density, how does the escape velocity scale with mass? (*Hint: if all planets have the same density, then you can write the radius of the planet in terms of its mass*).

a) v_E scales as $M^{1/2}$
b) v_E scales as $M^{1/3}$
c) v_E scales as M^3
d) v_E scales as $M^{3/2}$
e) v_E scales as M

The density ρ of an object is given by,

$$\rho = \frac{M}{\frac{4}{3}\pi R^3},$$

which is just its mass divided by its volume. We can solve for the radius and find,

$$R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}.$$

We can now put this in the equation for v_E above,

$$v_E = \sqrt{\frac{2GM}{R}} = \left(2GM\left[\frac{4\pi\rho}{3M}\right]^{1/3}\right)^{1/2} = \left(\frac{32\pi\rho G^3}{3}\right)^{1/6} M^{1/3} \propto M^{1/3}$$

So the escape velocity scales as the mass of the planet to the one-third power.

Answer b) v_E scales as $M^{1/3}$

Question 8 (10 points)

Astronomers estimate that Rocky has a mass that is only one eighth of the mass of the Earth. Assume he has the same density as the Earth. What is his escape velocity relative to that of the Earth?

a) half of the escape velocity of the Earth

b) one eighth of the escape velocity of the Earth
c) one fourth of the escape velocity of the Earth
d) the same as the escape velocity of Earth

We just derived that the escape velocity scales as the one-third power of the mass for objects of the same density. Therefore, we can write:

$$\frac{(v_E)_{George}}{(v_E)_{Earth}} = \left(\frac{M_{George}}{M_{Earth}}\right)^{1/3} = \left(\frac{1}{8}\right)^{1/3} = 0.5 \, .$$

Therefore, the escape velocity at Rocky's surface is 50%, or half, of the escape velocity at the surface of the Earth.

Answer a) half of the escape velocity of the Earth

In class I said that the thermal velocity of a gas of temperature T with composed of molecules or atoms of mass m is given by:

$$v = \sqrt{\frac{2kT}{m}}$$

Where k is a constant.

Question 9 (10 points)

For fixed particle mass m, how does the thermal velocity scale with distance from the Sun?

a) the thermal velocity scales as $d^{-1/2}$.
c) the thermal velocity scales as $d^{1/2}$.
d) the thermal velocity scales as $d^{-1/4}$.
e) the thermal velocity scales as $d^{-3/2}$.

We have derived that the equilibrium temperature scales as the distance from the Sun to the negative one-half power, i.e.,

$$T = \left(\frac{L}{16\pi\sigma}\right)^{1/4} d^{-1/2} \, .$$

We can substitute this temperature into the expression for the thermal velocity above,

$$v = \sqrt{\frac{2kT}{m}} = \left[\frac{2k}{m}\left(\frac{L}{16\pi\sigma}\right)^{1/4} d^{-1/2}\right]^{1/2} = \left(\frac{k^4 L}{\pi\sigma m^4}\right)^{1/8} d^{-1/4}.$$

Therefore, the thermal velocity scales as the distance to the negative one-fourth power.

Answer d) the thermal velocity scales as d$^{-1/4}$

Question 10 (10 points)

If the atmosphere of Rocky is composed of gas particles of the same mass as the atmosphere of the Earth, what is the thermal velocity of the gas particles in Rocky's atmosphere relative to those in the Earth's atmosphere?

a) the thermal velocity is the same as the Earth's.
b) the thermal velocity is half of the Earth's.
c) the thermal velocity is about a fourth of the Earth's.
d) the thermal velocity is about twice the Earth's.

Since the thermal velocity scales as the distance to the negative one-fourth power, we can write:

$$\frac{v_{George}}{v_{Earth}} = \left(\frac{d_{George}}{d_{Earth}}\right)^{-1/4} = (16)^{-1/4} = 0.5.$$

Answer b) the thermal velocity is half of the Earth's.

Question 11 (5 points)

Given what you know about his mass, radius, and temperature, is Rocky capable of maintaining an atmosphere like the Earth's?

a) Yes
b) No

We have calculated that both the thermal velocity and escape speed of Rocky are the same fraction (50%) of the Earth's escape speed and thermal velocity. Since the ability to maintain an atmosphere is given by the balance between these two speeds, we can conclude that (for the same particle mass *m*), Rocky can maintain the same atmosphere as the Earth.

Answer a) Yes