1 Monday, November 28: Comptonization

When photons and electrons coexist in the same volume of space, their scattering interactions can transfer energy from photons to electrons (Compton scattering) or from electrons to photons (inverse Compton scattering). In general, therefore, when photons travel through a region containing free electrons, their spectrum will be changed. In other words, the shape of the specific intensity I_ν will be modified as photons are scattered to lower or higher ν. The change in the spectrum of light due to scattering from electrons is referred to as *Comptonization*.

If light passes through a blob of material of finite size, the average change in a photon's energy is given by the Compton y parameter. The magnitude of y is given by the relation

$$y = \frac{\Delta\epsilon}{\epsilon} \times N_{\rm es} \ , \tag{1}$$

where $\Delta\epsilon/\epsilon$ is the average fractional change in the photon's energy $\epsilon = h\nu$ from a single scattering, and $N_{\rm es}$ is the average number of scatterings from electrons as the photon passes through the medium. If $y \ll 1$, then the spectrum of light will only be slightly changed as it passes through the medium. If $y \gg 1$, however, the spectrum can be strongly modified.

Computing $N_{\rm es}$ is fairly easy. Suppose that the medium through which the light passes has a diameter L. The number density of free electrons in the medium is n_e, and the typical Lorentz factor of the electrons is γ. If $\epsilon \ll m_e c^2/\gamma$, as we saw last Wednesday, the photon-electron interactions can be treated as Thomson scattering in the electron's rest frame. In that case, the mean free path of the photons will be $\ell_{\rm es} = 1/(n_e \sigma_T)$, where σ_T is the Thomson cross-section of the electron. The optical depth of the blob is then

$$\tau_{\rm es} = L/\ell_{\rm es} = n_e \sigma_T L \ . \tag{2}$$

If $\tau_{\rm es} \ll 1$, the average number of scatterings will be $N_{\rm es} = \tau_{\rm es} \ll 1$. If $\tau_{\rm es} \gg 1$, the photon will random-walk through the medium, traveling an rms distance

$$\langle R^2 \rangle^{1/2} \approx \sqrt{N} \ell_{\rm es} \tag{3}$$

after N scatterings. Traversing the medium requires traveling a distance $\langle R^2 \rangle^{1/2} \approx L$, implying

$$N_{\rm es} \approx (L/\ell_{\rm es})^2 \approx \tau_{\rm es}^2 \ . \tag{4}$$

A good approximation for $N_{\rm es}$ for all optical depths is

$$N_{\rm es} = \max(\tau_{\rm es}, \tau_{\rm es}^2) \ . \tag{5}$$

For optically thick media, with $\tau_{\rm es} = n_e \sigma_T L \gg 1$, the photons undergo a great many scatterings before emerging from the medium.

Computing $\Delta\epsilon/\epsilon$ is a little more complicated, since it depends on the distribution of electron energies. One of the most useful cases is when the light passes through an ionized gas in which the electrons have a non-relativistic thermal distribution, characterized by a temperature $T < m_e c^2/k \sim 6 \times 10^9\,{\rm K}$. In this case, the average fractional energy change of a photon undergoing a single scattering is

$$\left.\frac{\Delta\epsilon}{\epsilon}\right|_{\rm NR} = \frac{4kT - \epsilon}{m_e c^2} \ . \tag{6}$$

When $4kT \gg \epsilon$, and the electron's thermal energy is much larger than the initial photon energy, the photon gains energy on average. When $4kT \ll \epsilon$, and the electron's thermal energy is much smaller than the initial photon energy, the photon loses energy on average. The break-even point is at $\epsilon = 4kT$.[1] As an example, let's consider the Sunyaev-Zel'dovich effect, in which CMB photons pass through the hot intracluster gas of a rich cluster of galaxies. In this case, $4kT \gg \epsilon$, and the Compton parameter for the photons passing through the cluster is[2]

$$y_{\rm NR} \approx \frac{4kT}{m_e c^2} \max(\tau_{\rm es}, \tau_{\rm es}^2) \ . \tag{7}$$

For CMB photons passing through an optically thin cluster, the Compton y parameter for the Sunyaev-Zel'dovich effect is

$$y_{\rm SZ} \approx \frac{4kT}{m_e c^2} n_e \sigma_T L \sim 4 \times 10^{-5} \left(\frac{T}{10^7\,{\rm K}}\right) \left(\frac{n_e}{10^{-3}\,{\rm cm}^{-3}}\right) \left(\frac{L}{3\,{\rm Mpc}}\right) \ . \tag{8}$$

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[1] The factor of 4 comes from considering the case in which the electrons and photons come to thermal equilibrium at temperature T. In that case $\Delta\epsilon/\epsilon$ must equal zero.

[2] You may recall that on November 21, we computed $\Delta\epsilon/\epsilon \sim (kT/m_e c^2)^{1/2}$ for Sunyaev-Zel'dovich scattering, rather than the correct (and smaller) value $\Delta\epsilon/\epsilon \sim kT/m_e c^2$. This is because I assumed that all collisions were head-on, and the photon was always scattered back the way it came. This produces the maximum possible energy transfer, and hence leads to an overestimate of $\Delta\epsilon/\epsilon$.

Even for a big cluster, we expect $y_{\rm SZ} \ll 1$.

In general, when low-energy photons ($\epsilon \ll m_e c^2$) interact with non-relativistic thermal electrons ($kT \ll m_e c^2$), the fractional energy change in a single scattering is small. This means that the evolution of the spectrum of photons can be solved using perturbation techniques. Let $n(\vec{x}, \vec{p}) d^3x d^3p$ be the number of photons in a volume element d^3x at position $\vec{x}$ that have momenta in a momentum element d^3p at momentum $\vec{p}$. The function $n(\vec{x}, \vec{p})$ is called the *phase space density*; those of you who have read 'Galactic Dynamics' by Binney and Tremaine probably have phase space densities permanently embedded in your brains. Just for practice, let's compute the phase space density of blackbody radiation with a Planck spectrum of energies. Blackbody radiation has an energy per unit volume per unit frequency of

$$u_\nu d^3x d\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{\exp(\frac{h\nu}{kT_{\rm rad}}) - 1} d^3x d\nu \ , \tag{9}$$

and thus a photon number per unit volume per unit frequency of

$$n_\nu d^3x d\nu = \frac{u_\nu}{h\nu} = \frac{8\pi \nu^2}{c^3} \frac{1}{\exp(\frac{h\nu}{kT_{\rm rad}}) - 1} d^3x d\nu \ . \tag{10}$$

For isotropic blackbody radiation, the basic momentum element can be written as

$$4\pi p^2 dp = \frac{4\pi h^3}{c^3} \nu^2 d\nu \ , \tag{11}$$

since $p = h\nu/c$. Then by setting

$$n_\nu d^3x d\nu = n(\vec{x}, \vec{p}) d^3x d^3p = n(\vec{x}, \vec{p}) d^3x d\nu \left(\frac{4\pi h^3}{c^3} \nu^2 \right) \ , \tag{12}$$

we find that the phase space density of blackbody photons is

$$n(\vec{x}, \vec{p}) = \frac{c^3}{4\pi h^3 \nu^2} n_\nu = \frac{2}{h^3} \frac{1}{\exp(\frac{h\nu}{kT_{\rm rad}}) - 1} \ , \tag{13}$$

which is the famous Bose-Einstein distribution. Above some characteristic photon energy $h\nu \sim kT_{\rm rad}$, corresponding to a photon momentum $h\nu/c \sim kT_{\rm rad}/c$, the number of photons falls off exponentially.

The equation that tells how $n(\vec{x}, \vec{p})$ evolves in the presence of inverse Compton scattering from non-relativistic thermal electrons is called the *Kompaneets equation*, after the Soviet scientist A. S. Kompaneets, who derived it in the mid-twentieth century. The derivation of the Kompaneets equation has been called "distinctly non-trivial". I will simply write it down for your general cultural enrichment. The equation is simpler in appearance if we use some dimensionless units. Let $x = h\nu/(kT)$ be the photon energy in units of the thermal energy of the electrons (not their rest energy). In addition, $t_c = t(c/\ell_{\rm es}) = tcn_e\sigma_T$ is the time in units of the mean time between scatterings. We may then write the time evolution of the photon phase space density $n(\vec{x}, \vec{p})$ in the form

$$\frac{dn}{dt_c} = \left(\frac{kT}{m_e c^2}\right) \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 (\frac{\partial n}{\partial x} + n + n^2) \right] \ , \tag{14}$$

where T is the temperature of the non-relativistic electrons with which the photons are interacting. In general, as Rybicki and Lightman gently inform us, equation (14) is yet another differential equation that must be solved numerically.

There are a few situations in which the Kompaneets equation can be tackled analytically. Suppose you have a region filled with hot ionized gas at a temperature T. There is no magnetic field, so the main photon emission process is bremsstrahlung. The main photon absorption process is free-free absorption. The main photon scattering process is Thomson scattering. The Thomson scattering coefficient, as we know, is independent of frequency:

$$\alpha^{\rm es} = 1/\ell_{es} = n_e \sigma_T \ . \tag{15}$$

However, the free-free absorption coefficient, as we learned on November 4, is strongly dependent on frequency in the low frequency limit ($h\nu \ll kT$):

$$\alpha_\nu^{\rm ff} = 0.018 {\rm cm}^{-1} T^{-3/2} \frac{Z^2 n_e n_i}{\nu^2} \overline{g}_{\rm ff} \ , \tag{16}$$

where all quantities are in cgs units, and $\overline{g}_{\rm ff}$ is the Gaunt factor. Since the absorption coefficient is proportional to ν^{-2}, at frequencies lower than a critical value ν_0, absorption will dominate over scattering. For fully ionized hydrogen ($Z = 1$, $n_e = n_i$), the value of ν_0 is given by

$$\nu_0 = 1.6 \times 10^{11} \,{\rm Hz} T^{-3/4} n_e^{1/2} \overline{g}_{\rm ff}^{1/2} \ . \tag{17}$$

When $\nu \ll \nu_0$, the spectrum is unmodified by scattering; it will just be the standard bremsstrahlung spectrum cut off by free-free absorption at low

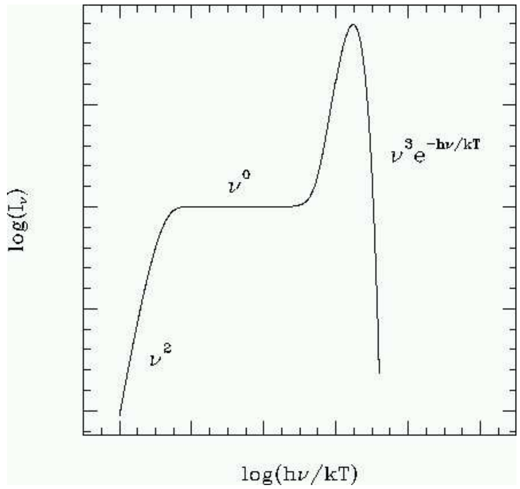

Figure 1: Spectrum of hot gas, showing the effects of saturated inverse Compton scattering at high frequencies.

frequencies (as seen in Figure 1). However, when the gas temperature is $kT \gg h\nu_0$, we expect most of the photons produced to have $\nu > \nu_0$, and to be significantly affected by scattering as well as free-free absorption. If the Compton y parameter is $y \gg 1$, then the spectrum at high frequencies will be significantly modified by inverse Compton scattering, creating a lack of photons at frequencies $\nu > kT/h$. The more detailed calculations in the text reveal that there will be a "Compton hump" created at the frequency $\nu = 3kT/h$ (Figure 1).

2 Wednesday, November 30: Plasma Effects

Suppose that a linearly polarized, monochromatic plane wave is propagating along the x axis, so that

$$\vec{E} = E_0 \hat{e}_y \cos(kx - \omega t) \ . \tag{18}$$

If the wave is propagating through a vacuum, it travels with speed $c = \omega/k$, its amplitude E_0 is constant, and it maintains its linear polarization, with $\vec{E}$ always pointing in the z direction (or the $-z$ direction, depending on phase). However, if the wave *isn't* propagating through a vacuum, its speed, amplitude, and polarization can all be altered.

Suppose, first of all, that the wave is traveling through ionized gas (otherwise known as *plasma*). The number density of free electrons is n_e, and the number density of positive ions is sufficient to maintain charge neutrality. (For the moment, to keep things simple, let's assume that there's no external magnetic field.) The changing electromagnetic field due to the traveling wave makes the free electrons accelerate. Since the electron charge is $q = -e$, the equation of motion is

$$m_e \dot{\vec{v}} = -e\vec{E} \tag{19}$$

in the non-relativistic limit, where $v \ll c$. The velocity of the electrons will thus be

$$\vec{v} = \frac{eE_0 \hat{e}_y}{\omega m_e} \sin(kx - \omega t) \ , \tag{20}$$

where I've chosen the constant of integration so that the time averaged electron velocity is zero. The current density resulting from the electron's motion is

$$\vec{j} = -n_e e \vec{v} \ . \tag{21}$$

The massive, slow-moving ions don't contribute significantly to the current, so we may ignore them. The oscillatory electron motions set up by the plane wave produce an alternating current:

$$\vec{j} = -\frac{n_e e^2 E_0 \hat{e}_y}{\omega m_e} \sin(kx - \omega t) \ . \tag{22}$$

Note that the current $\vec{j}$ is $90°$ out of phase with $\vec{E}$ and has a maximum value

$$j_{\max} = \sigma E_0 \ , \tag{23}$$

where σ is the *electrical conductivity*,

$$\sigma = \frac{n_e e^2}{\omega m_e} \ . \tag{24}$$

The larger the value of σ, the greater the current set up by a given electric field $\vec{E}$. For plane waves traveling through a plasma, the conductivity is greatest for low-frequency waves.

Now, when we solve Maxwell's Equations for a plane wave traveling through a plasma, we must include a current density j_q resulting from the acceleration of the free electrons. With this current, the dispersion relation for the plane wave becomes

$$c^2 k^2 = \epsilon \omega^2 \ , \tag{25}$$

where the factor ϵ is the *dielectric constant*, and has the value

$$\epsilon = 1 - \frac{4\pi\sigma}{\omega m_e} = 1 - \frac{4\pi n_e e^2}{\omega^2 m_e^2} \ . \tag{26}$$

Note that $\epsilon \leq 1$, and is equal to unity only in the limit $n_e \to 0$ or $\omega^2 \to \infty$. The value of the dielectric constant is frequently written in the form

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} \ , \tag{27}$$

where ω_p is called the *plasma frequency*, and has the value

$$\omega_p = \left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2} \approx 5.6 \times 10^4\,\mathrm{s}^{-1} \left(\frac{n_e}{1\,\mathrm{cm}^{-3}}\right)^{1/2} \ . \tag{28}$$

The plasma frequency is the frequency of oscillations that would be set up if all the electrons were displaced by a small distance $\delta\vec{x}$ with respect to the positively charged ions.

For angular frequencies $\omega < \omega_p$, the wavenumber

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2} \tag{29}$$

is *imaginary*, leading to an exponentially decaying wave amplitude. The plasma frequency represents an angular frequency below which waves cannot stably propagate through an ionized gas. The Local Bubble, and similar regions of hot ionized gas in the interstellar medium, has $n_e \sim 10^{-2}\,\mathrm{cm}^{-3}$, implying a plasma frequency $\omega_p \sim 6000\,\mathrm{s}^{-1}$. Thus, radio waves with $\nu < \omega_p/(2\pi) \sim 1\,\mathrm{kHz}$ cannot propagate through the hottest portions of the interstellar medium. Instead, they are *evanescent*, in the jargon of plasma physicists. This means that the attenuation length over which waves are damped, $\ell \sim 2\pi c/\omega_p$, is short compared to the wavelength $\lambda = 2\pi c/\omega$. In the Local Bubble, where $\omega_p \sim 6 \times 10^3\,\mathrm{s}^{-1}$, the attenuation length is a trifling $\ell \sim 300\,\mathrm{km}$.[3]

[3] This is yet another reason why radio astronomers don't observe at Extremely Low Frequencies. Electromagnetic waves with $\nu \ll 1\,\mathrm{kHz}$ are strongly attenuated in the Local Bubble.

The speed of light through a plasma is not going to be equal to c. Suppose that a plane wave of angular frequency ω is propagating through the plasma, so that

$$\vec{E} = E_0 \hat{e}_y \cos(kx - \omega t) \ . \tag{30}$$

If we choose a particular wavecrest – for instance, the one for which $kx - \omega t = 2n\pi$ – the speed with which this crest travels is given by the relation

$$k \frac{dx}{dt} - \omega = 0 \ , \tag{31}$$

or

$$\frac{dx}{dt} = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}} \ . \tag{32}$$

This speed, $v_{ph} \equiv \omega/k$, is known as the *phase velocity*, since it is the speed with which a point of fixed phase on a sinusoidal wave travels through space.

Notice that $v_{ph} > c$ for light traveling through a plasma with $\omega > \omega_p$. Does this mean we should be shocked and alarmed? No, don't panic; a phase velocity greater than c doesn't violate special relativity. As a wavecrest, or a point of any other phase, travels along at $v_{ph} > c$, it doesn't carry any information with it, and thus cannot be used to transmit information at superluminal speeds. In order to use an electromagnetic wave to carry information, you must *modulate* the wave; that is, you have to modify it in some way from a monochromatic wave of constant amplitude and constant linear polarization. You can, for instance, use *frequency* modulation (like FM radio) or *amplitude* modulation (like AM radio).[4] Suppose you create an amplitude modulated signal (Figure 2) in the form of a compact wave packet. This can be done by adding together many sinusoidal waves of slightly different angular frequency. The speed with which the wave packet travels is the *group velocity*

$$v_g \equiv \frac{\partial \omega}{\partial k} \ . \tag{33}$$

For waves traveling through a plasma, taking the derivative of the dispersion relation (equation 29) yields

$$1 = \frac{1}{2c}(\omega^2 - \omega_p^2)^{-1/2} 2\omega \frac{\partial \omega}{\partial k} \ , \tag{34}$$

[4] You can also use polarization modulation, but PM radio doesn't seem to have caught on commercially.

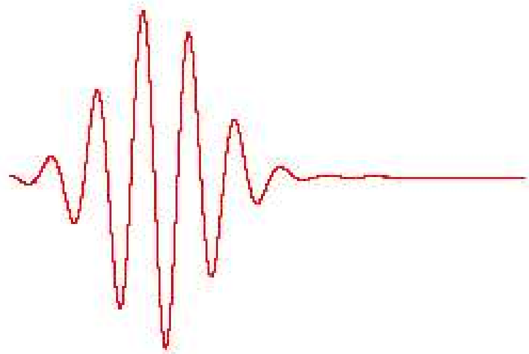

Figure 2: Amplitude modulated wave packet.

or

$$v_g = \frac{\partial\omega}{\partial k} = \frac{c(\omega^2 - \omega_p^2)^{1/2}}{\omega} = c\sqrt{1 - \omega_p^2/\omega^2} \ . \tag{35}$$

The group velocity is always less than or equal to c for waves with $\omega > \omega_p$.

The group velocity $v_g(\omega)$ is greatest for the highest frequency component of a signal. If an astronomical source emits pulses of light, then the highest frequency component of each pulse will reach us first, while the lowest frequency components lag behind (or are damped entirely, in the case $\omega < \omega_p$). Consider, as an example, a pulsar at a distance d from the Earth. The time it takes for light of angular frequency ω to reach the Earth is

$$t_p(\omega) = \int_0^d \frac{ds}{v_g(\omega)} = \frac{1}{c}\int_0^d \frac{ds}{\sqrt{1 - \omega_p(s)^2/\omega^2}} \ . \tag{36}$$

Since the free electron density n_e usually varies along the line of sight to the pulsar, the plasma frequency $\omega_p \propto n_e^{1/2}$ does as well. For frequencies much greater than ω_p, we may write

$$\frac{1}{\sqrt{1 - \omega_p^2/\omega^2}} \approx 1 + \frac{\omega_p^2}{2\omega^2} \ , \tag{37}$$

and thus

$$t_p(\omega) \approx \frac{d}{c} + \frac{1}{2c\omega^2}\int_0^d \omega_p^2 ds \ . \tag{38}$$

Suppose you measure the arrival time of a pulse at two different frequencies, ω_1 and ω_2, with $\omega_2 > \omega_1$. The higher frequency component will arrive first, followed by the lower-frequency component, after a lapse of time

$$\Delta t \equiv t_p(\omega_1) - t_p(\omega_2) \approx \frac{1}{2c}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)\int_0^d \omega_p^2 ds \ . \tag{39}$$

Using the relation $\omega_p^2 = (4\pi e^2/m_e)n_e$, we find that

$$\Delta t = \frac{2\pi e^2}{m_e c}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)\int_0^d n_e ds \ . \tag{40}$$

The integral of n_e along the line of sight to the pulsar is known as the *dispersion measure* (DM) of the pulsar. By measuring the time delay between the arrival time of a pulse at two different frequencies, the dispersion measure can be measured. The Crab pulsar, for instance, has a dispersion measure of

$$\mathrm{DM}_{\mathrm{crab}} = \int_0^{d_{\mathrm{crab}}} n_e ds = 57\,\mathrm{pc\,cm^{-3}} \ . \tag{41}$$

Since we know the distance to the Crab is $d_{\mathrm{crab}} \approx 2000\,\mathrm{pc}$, we can compute that the average density of free electrons between us and the Crab pulsar is

$$\langle n_e \rangle = \frac{\mathrm{DM}_{\mathrm{crab}}}{d_{\mathrm{crab}}} \approx \frac{57\,\mathrm{pc\,cm^{-3}}}{2000\,\mathrm{pc}} \approx 0.03\,\mathrm{cm^{-3}} \ . \tag{42}$$

For pulsars whose distance is unknown, you can estimate the distance by assuming that the average electron density along the line of sight is $n_e \approx 0.03\,\mathrm{cm^{-3}}$, but this is fraught with uncertainty, given the inhomogeneity of the interstellar gas.

3 Friday, December 2: Faraday Rotation (and Cherenkov Radiation)

In the previous lecture, I discussed the propagation of light through a plasma, in the absence of an external magnetic field. In that case, the only special frequency is the plasma frequency,

$$\omega_p \equiv \left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2} \approx 9700\,\mathrm{s^{-1}}\left(\frac{n_e}{0.03\,\mathrm{cm^{-3}}}\right)^{1/2} \ . \tag{43}$$

If there exists a constant magnetic flux density $\vec{B}_0$ within the plasma, there will be an additional frequency of interest: the cyclotron frequency,

$$\omega_{\rm cyc} \equiv \frac{eB_0}{m_e c} \approx 17\,{\rm s}^{-1} \left(\frac{B_0}{10^{-6}\,{\rm G}} \right) . \tag{44}$$

(I've scaled the plasma frequency and the cyclotron frequency to typical values of n_e and B_0 you might find in the interstellar medium.) The presence of $\vec{B}_0$ also introduces anisotropy into the problem. Suppose that $\vec{B}_0 = B_0 \vec{e}_x$; if we locate ourselves at large x, so that $\vec{B}_0$ is pointing toward us, we will see non-relativistic electrons orbiting in a *counterclockwise* direction at the cyclotron frequency. Thus, the magnetic field creates a preferred sense of orbital direction for the electrons, as well as a preferred orbital frequency, $\omega_{\rm cyc}$.

The effects of the magnetic field $\vec{B}_0 = B_0 \vec{e}_x$ on the propagation of light are seen most clearly if we look at *circularly polarized* light traveling along the magnetic flux vector $\vec{B}_0$. As we've already seen, circularly polarized light can be created by superimposing two linearly polarized waves that are 90 degrees out of phase. For instance,

$$\vec{E} = E_0 [\hat{e}_y \cos(kx - \omega t) \pm \hat{e}_z \sin(kx - \omega t)] \tag{45}$$

is a circularly polarized wave traveling in the positive x direction. The two choices of sign in equation (45) correspond to the two varieties of circular polarization. If an observer who sees the wave propagating toward him detects counterclockwise rotation of $\vec{E}$ at fixed x, the polarization is *left-handed.* If the observer detects clockwise rotation of $\vec{E}$, the polarization is *right-handed.*

The equation of motion for a non-relativistic electron, accelerated both by the circularly polarized wave and by the magnetic field $\vec{B}_0$, is

$$m_e \dot{\vec{v}} = -e [\vec{E} + \frac{\vec{v}}{c} \times \vec{B}_0] . \tag{46}$$

Since we know $\vec{E}$ and $\vec{B}_0$, we can solve this differential equation to yield $\vec{v}$, the velocity of the electron. I will leave it as an exercise for the reader to demonstrate that

$$\vec{v} = -\frac{eE_0}{m_e(\omega \pm \omega_{\rm cyc})} [\hat{e}_y \sin(kx - \omega t) \mp \hat{e}_z \cos(kx - \omega t)] . \tag{47}$$

The electron moves on a circular orbit (counterclockwise for for left polarization, clockwise for right polarization). The orbital velocity depends on

$\omega_{\rm cyc}$ as well as on ω, and is *not* the same for right polarized waves as for left polarized waves.

A right polarized wave produces a maximum current density

$$j_R = -e n_e v_R = \frac{e^2 n_e E_0}{m_e(\omega + \omega_{\rm cyc})} \ , \tag{48}$$

while a left polarized wave produces a maximum current density

$$j_L = -e n_e v_L = \frac{e^2 n_e E_0}{m_e(\omega - \omega_{\rm cyc})} \tag{49}$$

which is larger than j_R for $\omega > \omega_{\rm cyc} > 0$. (Note the resonance for left polarized waves at $\omega = \omega_{\rm cyc}$.) The difference in current for the two types of circular polarization results in a difference in dielectric constant, and consequently a difference in the phase velocity. In the limit $\omega \gg \omega_{\rm cyc}$ and $\omega \gg \omega_p$,

$$v_{ph}(R) \approx c\left[1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2} - \frac{1}{2}\frac{\omega_p^2\omega_{\rm cyc}}{\omega^3}\right] \tag{50}$$

$$v_{ph}(L) \approx c\left[1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2} + \frac{1}{2}\frac{\omega_p^2\omega_{\rm cyc}}{\omega^3}\right] \ . \tag{51}$$

The difference in phase velocity for right and left polarized light is small; if you are observing radio signals through the interstellar medium of our galaxy, you would expect

$$\Delta v \equiv v_{ph}(L) - v_{ph}(R) \approx c\frac{\omega_p^2\omega_{\rm cyc}}{\omega^3} \tag{52}$$

$$\sim 2 \times 10^{-10}\,{\rm cm\,s^{-1}}\left(\frac{\nu}{1\,{\rm GHz}}\right)^{-3} \ , \tag{53}$$

for the values of ω_p and $\omega_{\rm cyc}$ quoted earlier. Although the difference is small, it has observable consequences.

A linearly polarized wave is the sum of a right circularly polarized wave and a left circularly polarized wave. The orientation of the plane of polarization depends on the relative phases of the two waves. As the left polarized wave slowly pulls ahead of the right polarized wave, the relative phases shift and the plane of polarization rotates. This rotation is called *Faraday rotation*, and can be significant for astronomical sources. During a short time

Δt, the right polarized wave pulls ahead by a distance $\Delta x = \Delta v \Delta t$, and the plane of polarization rotates by an angle

$$\begin{aligned} \Delta\theta &= \pi \frac{\Delta v \Delta t}{\lambda} \approx \pi \left(c \frac{\omega_p^2 \omega_{\rm cyc}}{\omega^3} \right) \frac{\omega}{2\pi c} \Delta t & (54) \\ &\approx \frac{\omega_p^2 \omega_{\rm cyc}}{2\omega^2} \Delta t \; . & (55) \end{aligned}$$

If we integrate along the line of sight from a radio source at distance d, the total rotation of the plane of polarization is

$$\theta \approx \frac{1}{2\omega^2} \int_0^d \omega_p^2 \omega_{\rm cyc} \frac{ds}{c} \; . \quad (56)$$

Substituting for ω_p and $\omega_{\rm cyc}$, we find

$$\theta \approx \frac{2\pi e^3}{m_e^2 c^2 \omega^2} \int_0^d n_e B_0 ds \; , \quad (57)$$

where B_0 is the component of the magnetic field that lies along the line of sight.

Suppose you look at a synchrotron source, such as a pulsar, at two different angular frequencies, ω_1 and ω_2. By measuring the arrival time of pulses at the two frequencies, you can compute the dispersion measure,

$$\mathrm{DM} \equiv \int_0^d n_e ds \; , \quad (58)$$

as we saw in the previous lecture. By measuring the orientation of the polarization at the two frequencies, you can compute the *rotation measure*, usually defined as[5]

$$\mathrm{RM} \equiv \frac{e^3}{2\pi m_e^2 c^4} \int_0^d n_e B_0 ds \; . \quad (59)$$

Defined in this way, the rotation measure is the rotation θ of the polarization divided by the square of the wavelength of observation, $\lambda^2 = (2\pi c/\omega)^2$. Knowing both the DM and the RM, you can compute an average magnetic flux density $\langle B_0 \rangle$ along the line of sight:

$$\langle B_0 \rangle = \frac{2\pi m_e^2 c^4}{e^3} \frac{\mathrm{RM}}{\mathrm{DM}} = 3.82 \times 10^{16}\,\mathrm{G} \frac{\mathrm{RM}}{\mathrm{DM}} \; , \quad (60)$$

[5] No, I don't know who defined it in this way, but it's what you'll find when you look through the literature.

when all quantities are in cgs units. For instance, according to Lang's "Astrophysical Data", the Crab pulsar has a dispersion measure

$$\mathrm{DM_{crab}} = 56.791\,\mathrm{pc\,cm^{-3}} = 1.753 \times 10^{20}\,\mathrm{cm^{-2}} \tag{61}$$

and rotation measure[6]

$$\mathrm{RM_{crab}} = -42.3\,\mathrm{rad\,m^{-2}} = -4.23 \times 10^{-3}\,\mathrm{cm^{-2}}\ . \tag{62}$$

This yields an average magnetic field

$$\langle B_0 \rangle = 3.82 \times 10^{16}\,\mathrm{G} \left(\frac{-4.23 \times 10^{-3}\,\mathrm{cm^{-2}}}{1.753 \times 10^{20}\,\mathrm{cm^{-2}}} \right) = -9.22 \times 10^{-7}\,\mathrm{G} \tag{63}$$

along our line of sight to the Crab. Similar calculations for other pulsars also tend to yield magnetic fields at the microgauss level.

While we are talking about ionized gas, I want to say a few words about Cherenkov[7] radiation. Since the group velocity v_g in an unmagnetized plasma is

$$v_g = c\sqrt{1 - \omega_p^2/\omega^2} < c\ , \tag{64}$$

it is possible for a highly relativistic charged particle, such as a cosmic ray proton, to travel with a speed $v > v_g$. Such a particle produces Cherenkov radiation by a process similar to the production of a sonic boom by a supersonic particle. If the charged particle is moving with a velocity $v > v_g$, it forms a conical "photonic shockwave" at an angle θ_C with respect to the particle's direction of motion (Figure 3). The Cherenkov angle θ_C is given by the relation

$$\cos\theta_C = v_g/v \approx \frac{\sqrt{1 - \omega_p/\omega^2}}{\beta}\ , \tag{65}$$

where $\beta = v/c$ is the dimensionless velocity of the particle. Photons are emitted with trajectories perpendicular to the conical shockwave.

The group velocity in a plasma increases from 0 to c as the frequency of light goes from ω_p to ∞. Thus, for a given particle velocity v, there exists a

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[6]If I have the sign convention correct, a negative rotation measure means that $\vec{B}_0$ is pointing toward us, on average.

[7]You sometimes see the spelling "Cerenkov" or "Čerenkov". This is just a different transliteration for the original Cyrillic spelling.

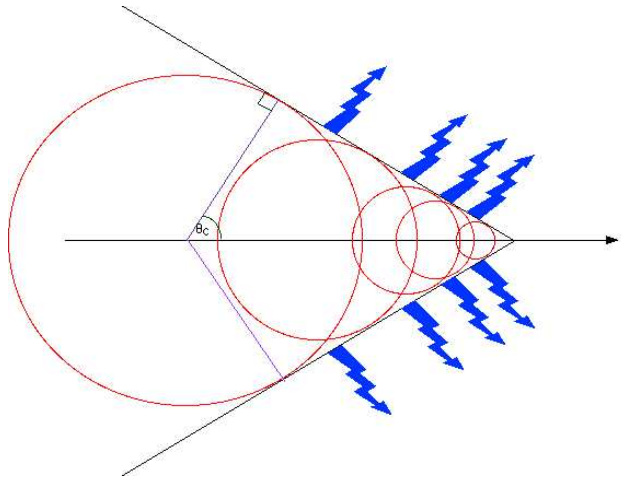

Figure 3: Geometry of Cherenkov radiation.

maximum possible frequency at which Cherenkov light can be emitted. This frequency is where $v_g = v$, or

$$\omega_{\max} = \frac{\omega_p}{\sqrt{1-\beta^2}} = \gamma\omega_p \ . \tag{66}$$

Thus, an ultrarelativistic particle traveling through an ionized gas can produce Cherenkov light with frequency much greater than the plasma frequency of the gas.