1 Monday, September 26: Radiative Transfer

As light travels through the universe, things happen to it. By interacting with charged particles, photons can gain energy; they can lose energy; they can change their direction of motion. Photons can also be absorbed by opaque lumps of matter, such as dust particles. The photons can also be joined by new photons emitted by the medium through which they travel.

It's very useful to have a shorthand description of what happens to the specific intensity I_ν of light as it propagates through the universe. This description is given by the equation of *radiative transfer*. The radiative transfer equation is a key equation for the study of stellar structure. (How does light get from the center of the Sun to its photosphere? Gamma rays are emitted by fusion reactions in the Sun's core, but the photons that escape from the photosphere are largely at near-infrared, visible, and near-ultraviolet frequencies. Obviously, as photons travel through the Sun, their mean energy must be decreased.) The radiative transfer equation is also a key equation for the study of the interstellar medium. (How does the light emitted by a star's photosphere differ from the light we observe at our telescope? The difference in absorption at different wavelengths can tell us about the composition of the dust and gas of the interstellar medium.)

To derive the equation of radiative transfer, let's start by setting up two identical transparent windows, each of area dA, separated by a short distance ds, as shown in Figure 1. After passing through the first window with specific intensity I_ν, the light passes through the second window with specific intensity I'_ν. If the space between the two windows is totally empty, then specific intensity is conserved:

$$I'_\nu = I_\nu \ . \tag{1}$$

This result is derived in the textbook; it results in a straightforward way from Euclidean geometry. Let me just note one of its implications. If you move further away from an extended light source (like the Sun, for instance), the power emitted per unit solid angle remains constant. However, the solid angle subtended by the light source decreases as one over the square of the distance. Thus, the light source's flux, integrated over its angular area, decreases as one over the square of the distance to the light source. The constancy of specific intensity implies the inverse square law of flux, and vice versa.

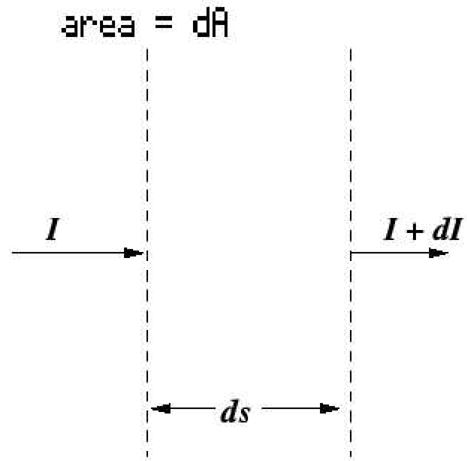

Figure 1: Geometry of radiative transfer

Now, suppose the space between the two windows isn't empty. It may contain hot, ionized gas (a bit of the Sun's interior, or at a much lower density, the coronal gas of the interstellar medium). It may contain cool molecular gas (a bit of the Earth's atmosphere, or at a much lower density, the molecular clouds of the interstellar medium). For that matter, it may contain a slab of lead. Each of these materials will scatter and absorb photons in different ways. They will also *produce* photons in different ways.

Let's look first at what happens if the material between the two windows is spontaneously emitting light. (That is, it's emitting photons regardless of the value of I_ν.) The gain in specific intensity going from the first window to the second is

$$dI_\nu = I'_\nu - I_\nu = j_\nu ds \ , \tag{2}$$

where j_ν is the *emission coefficient*, which has units of $\mathrm{erg\,s^{-1}\ cm^{-3}\,ster^{-1}\,Hz^{-1}}$ in cgs units. That is, if you had a cubic centimeter of the material between the windows, the emission coefficient is the power it would emit into a small solid angle $d\Omega$ in a small frequency interval $\nu \to \nu + d\nu$. (Note that the emission coefficient may be anisotropic.)

Although the spontaneous emission is independent of how much light is shining through the first window, the amount of absorption depends on I_ν. If 50% of a low-intensity light is absorbed in going through a filter, then 50% of a high-intensity light will be absorbed as well. for the absorption of light, therefore, we may write

$$dI_\nu = I'_\nu - I_\nu = -\alpha_\nu I_\nu ds \ , \tag{3}$$

where α_ν is the *absorption coefficient*, which has units of cm^{-1} in cgs units. The absorption coefficient represents the fraction of the specific intensity lost per centimeter traveled. It obviously can differ greatly for different media. The fact that we can see M31 at a distance of $d = 670\,\mathrm{kpc} \approx 2 \times 10^{24}\,\mathrm{cm}$ means that the average absorption coefficient along the line of sight to M31 must be $\alpha < d^{-1} \approx 5 \times 10^{-25}\,\mathrm{cm}^{-1}$ at visible frequencies. By contrast, the fact that I can't see through a piece of aluminum foil $2 \times 10^{-3}\,\mathrm{cm}$ thick means that its absorption coefficient must be $\alpha > d^{-1} \approx 500\,\mathrm{cm}^{-1}$ at visible frequencies.

The absorption coefficient is also frequency-dependent. Consider lead, for instance. Its absorption coefficient for gamma-rays with $E \approx 1\,\mathrm{MeV}$ is $\alpha_\nu \approx 0.6\,\mathrm{cm}^{-1}$. For lower-energy X-rays, though, with $E \approx 17\,\mathrm{keV}$, the absorption coefficient has the much higher value of $\alpha_\nu \approx 1400\,\mathrm{cm}^{-1}$. The absorption efficient can be either positive or negative in sign. If $\alpha_\nu > 0$, then energy is removed from the beam of light as it travels. if $\alpha_\nu < 0$, then energy is added to the beam by *stimulated emission* (as opposed to the spontaneous emission accounted for by the emission coefficient j_ν). There exist astronomical objects that act as *masers*, and have negative absorption coefficients at some frequencies. Mostly, though, we'll be dealing with objects that have positive absorption coefficients.

By combining the effects of spontaneous emission on the one hand, and the effects of absorption plus stimulated emission on the other hand, we find the radiative transfer equation:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \ . \tag{4}$$

Much of astronomy consists of finding appropriate values for the absorption coefficient α_ν and the emission coefficient j_ν, and then solving for I_ν as a function of position s.

Radiative transfer experts (and even some non-experts) frequently talk about the *optical depth* of some object. The optical depth is given the symbol τ_ν and is defined by the relation

$$d\tau_\nu \equiv \alpha_\nu ds \ . \tag{5}$$

Thus, the optical depth is a dimensionless number. For a path running from s_0 to s, the optical depth is

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')ds' \ . \tag{6}$$

A slab of material is called *optically thick* (or opaque) at a frequency ν when $\tau_\nu > 1$; it is called *optically thin* (or transparent) when $\tau_\nu < 1$. If the material happens to have an absorption coefficient α_ν that is constant in the region of interest, then

$$\tau_\nu = \alpha_\nu(s - s_0) \ . \tag{7}$$

The *mean free path* ℓ_ν is the distance $s - s_0$ for which $\tau_\nu = 1$. (That is, it's the thickness at which a slab of material goes from being transparent to opaque.) Again, if α_ν is constant,

$$\ell_\nu = \frac{1}{\alpha_\nu} \ . \tag{8}$$

For example, the mean free path of 1 MeV photons in lead is $\ell_\nu = 1/0.4\,\mathrm{cm}^{-1} = 2.5\,\mathrm{cm} \approx 1\,\mathrm{inch}$.

When the radiative transfer equation is divided by α_ν, it becomes

$$\frac{dI_\nu}{\alpha_\nu ds} = -I_\nu + \frac{j_\nu}{\alpha_\nu} \ , \tag{9}$$

or using the definition of optical depth,

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \ , \tag{10}$$

where $S_\nu \equiv j_\nu/\alpha_\nu$ is called the *source function.* It has the same units as the specific intensity ($\mathrm{erg\,s^{-1}\,cm^{-1}\,ster^{-1}\,Hz^{-1}}$).

So now we have a simple (deceptively simple!) equation that tells us how specific intensity varies as light travels through a medium. Our only difficulty is determining α_ν and j_ν, or alternatively S_ν and τ_ν, for every point along the light's path. Fortunately, there are a few interesting cases for which the solution is simple. Consider, for instance, the case when $S_\nu = j_\nu = 0$; that is, the medium through which the light travels doesn't glow spontaneously. In this case, the radiative transfer equation reduces to

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu \ , \tag{11}$$

with solution

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} \ . \tag{12}$$

The optical depth, we see, is just the dimensionless e-folding factor for absorption.

Another easy solution comes when the source function S_ν is constant in space. Then the solution of the radiative transfer equation is

$$I_\nu(\tau_\nu) = S_\nu + e^{-\tau_\nu}[I_\nu(0) - S_\nu] \ . \tag{13}$$

In the limit $\tau_\nu \to \infty$, $I_\nu \to S_\nu$. That is, if you are looking at an opaque slab of material, it doesn't really matter what light sources are on the other side of the slab. The specific intensity you see is dictated by the source function of the slab on the side facing you. This simplification provides a glimmer of hope for understanding emission from opaque objects (like stars, for instance, or lead bricks). We don't have to know the details of how photons are created within the region we can't see; we just need to know the source function S_ν in the region we can see (the photosphere of the star, for instance, or the surface of a lead brick).

2 Wednesday, September 28: Blackbody Radiation

Talking about the emission of light from opaque objects inevitably leads us to the topic of *blackbody radiation*. A blackbody, in the language of physics, is an object that absorbs every photon that strikes it. You might think it would be impossible to build an actual blackbody; after all, no material has an albedo of exactly zero at all wavelengths. However, it is possible to build a very good approximation of one. Start by making a closed box whose walls are made of a substance which is opaque at all frequencies of interest ($\tau_\nu \gg 1$). The interior of the box is empty.[1] We regulate the temperature of the box so that it has a constant temperature T (Figure 2). The walls of the box create photons, which bounce around inside the box, creating a photon gas which comes into thermal equilibrium with the walls of the box. The interior walls of the box now satisfy our definition of a blackbody!

[1]In practice, you don't have to pump down the interior to a high grade vacuum – as long as the gas inside is highly transparent ($\tau_\nu \ll 1$) at the frequencies of interest, that's a good enough approximation.

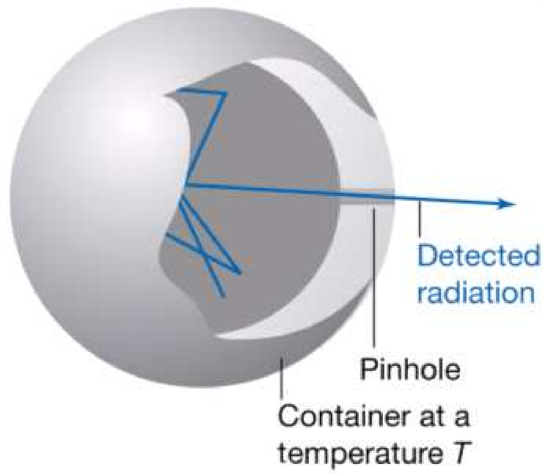

Figure 2: Creating blackbody radiation.

Although a photon may not be absorbed the first time it encounters the wall, or the second time, or the third time, eventually it will be absorbed. (After all, no material has an albedo of exactly one at any wavelength.) Now we make a tiny hole, no bigger than a pinhole, in the wall of the box.[2] When we place our eye (or other measuring instrument) to the hole, the specific intensity we measure, I_ν, will be that of blackbody radiation. The specific intensity of blackbody radiation is a function only of the temperature T. It is independent of the shape or size or chemical composition of the box. (If it weren't, then energy could flow between two blackbodies of the same temperature T but different shapes, sizes or chemical compositions. This would violate the laws of thermodynamics.) The specific intensity of blackbody radiation can thus be written in the form

$$I_\nu = B_\nu(T) \ , \tag{14}$$

where B_ν, known as the Planck function, is a function only of ν and T.

In the mid-nineteenth century, the form of $B_\nu(T)$ was poorly known.[3] However, experimental physicists began finding clues about the shape of B_ν. For instance, in 1879, Josef Stefan determined that the energy density inside

[2]If the window is very small, the leakage of photons through the hole won't significantly disturb the thermal equilibrium of the system.

[3]The physicist Gustav Kirchhoff wrote "It is a highly important task to find this universal function."

a blackbody cavity was

$$u(T) = aT^4 \ , \tag{15}$$

where $a = 7.6 \times 10^{-15}\,\mathrm{erg\,cm^{-3}\,K^{-4}}$. This law was independently discovered by Boltzmann, so it is known as the *Stefan-Boltzmann law*. This implies that the power emitted per unit area of a blackbody is

$$F(T) = \sigma T^4 \ , \tag{16}$$

where $\sigma = ac/4 = 5.7 \times 10^{-5}\,\mathrm{erg\,s^{-1}\,cm^{-2}\,K^{-4}}$.[4] In 1893, Wilhelm Wien noted that the frequency at which the specific intensity of a blackbody is maximized is directly proportional to the temperature:

$$\nu_{\rm max} = CT \ , \tag{17}$$

where $C = 5.9 \times 10^{10}\,\mathrm{Hz\,K^{-1}}$.[5] By the end of the 19th century, the shape of $B_\nu(T)$ was fairly well determined experimentally. The Planck function $B_\nu(T)$ is shown in Figure 3 for different values of T. At the low-frequency

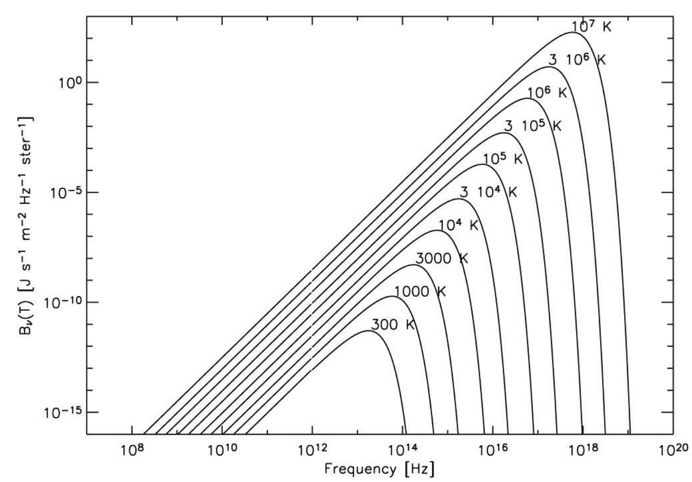

Figure 3: Planck function $B_\nu(T)$ for different values of T.

end ($h\nu \ll kT$), the Planck function is a powerlaw: $B_\nu \propto \nu^2$. This part

[4]You are an approximate blackbody with a temperature of $T = 310\,\mathrm{K}$. You therefore radiate $P = 5.2 \times 10^5\,\mathrm{erg\,s^{-1}} = 0.052$ watts from every square centimeter of your surface.

[5]With a temperature of $T = 310\,\mathrm{K}$, you therefore emit the most power at a frequency of $\nu_{\rm max} = 1.8 \times 10^{13}\,\mathrm{Hz}$, in the mid-infrared.

of the Planck function is called the *Rayleigh-Jeans* portion. At the high-frequency end ($h\nu \gg kT$), the Planck function has an exponential cutoff: $B_\nu \propto \nu^3 \exp(-h\nu/kT)$. This part of the Planck function is called the *Wien tail*. It seems that some bit of physics is suppressing the formation of high-frequency photons in a blackbody.

In the year 1900, given the observed properties of blackbody radiation, Max Planck tried to derive the functional form of $B_\nu(T)$. He first pointed out that a function of the form

$$B_\nu(T) \propto \frac{\nu^3}{\exp(h\nu/kT) - 1} \tag{18}$$

provided a smooth interpolation between the Rayleigh-Jeans (powerlaw) portion of B_ν and the Wien (exponential) tail. He then tried to find physical arguments for a function of the form given in equation (18). Planck's key realization was that light energy comes in *quanta*, each with energy $h\nu$. Thus, at a given frequency ν, the total photon energy inside a cavity must be $E = nh\nu$, where n is an integer. The probability of having a state of energy E is $P(E) \propto \exp(-E/kT)$, so if the energy E is quantized, the probability of having n quanta is

$$P(n\,\text{photons}) \propto \exp\left(-n\frac{h\nu}{kT}\right) . \tag{19}$$

Note this means that

$$\frac{P(n=1)}{P(n=0)} \propto \exp(-h\nu/kT) , \tag{20}$$

which is vanishingly small when $h\nu \gg kT$. The exponential cutoff in B_ν at high frequencies occurs because even a single high-frequency photon has an energy much greater than the characteristic energy kT of the system.

When you go through the complete derivation (as laid out in the text), the exact value of the Planck function is

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1} . \tag{21}$$

Deriving the Stefan-Boltzmann law and Wien's law from the Planck function is left as an exercise for the reader.

An object for which the source function S_ν is equal to the Planck function $B_\nu(T)$ is emitting *thermal radiation*. An object for which the specific intensity I_ν is equal to $B_\nu(T)$ is emitting *blackbody radiation*. To see the difference between thermal radiation and blackbody radiation, consider looking at a slab of material with optical depth τ_ν that is producing thermal radiation ($S_\nu = B_\nu(T)$). If no light is falling on the backside of the slab, then the specific intensity that we measure is (see yesterday's notes)

$$I_\nu = B_\nu(T)(1 - e^{-\tau_\nu}) \ . \tag{22}$$

If the slab is optically thick ($\tau_\nu \gg 1$), then

$$I_\nu \approx B_\nu(T) \tag{23}$$

and we observe blackbody radiation. If the slab is optically thin ($\tau_\nu \ll 1$) then

$$I_\nu \approx \tau_\nu B_\nu(T) \ll B_\nu(T) \ . \tag{24}$$

Since the optical depth τ_ν is a function of frequency, the spectrum that we see from an optically thin thermally radiating slab will not be a blackbody spectrum.

3 Friday, September 30: Temperature (and a Little Scattering)

The Planck function is so tremendously useful that astronomers frequently apply it in cases where it doesn't, strictly speaking, apply. The Cosmic Microwave Background (seen in last week's notes) is the outstanding example of a case where the Planck function really does fit the observed specific intensity. Stars, in many cases, are moderately well fit by a Planck spectrum. Hot gas clouds are well fit by Planck spectrum at the frequencies for which $\tau_\nu \gg 1$; usually the low-frequency end of the spectrum, for which $B_\nu \propto \nu^2$. If every object in the universe radiated like a perfect blackbody, then knowing the temperature T would tell you everything you needed to know about its specific intensity I_ν, and vice versa. Since objects are not perfectly blackbodies, the different methods we use to estimate their temperature yield divergent results. For a distance glowing object, we can estimate the temperature T by giving the brightness temperature T_b, the effective temperature $T_{\rm eff}$, or the color temperature T_c.

The *brightness temperature* T_b is actually an alternative way of stating the specific intensity I_ν of an object at a frequency ν. That is, $T_b(\nu)$ is the temperature for which the observed specific intensity I_ν obeys the equality

$$I_\nu = B_\nu(T_b) \ . \tag{25}$$

At the Raleigh-Jeans end of the Planck function, where where $h\nu \ll kT_b$, the brightness temperature is given by the relation

$$I_\nu = \frac{2\nu^2}{c^2} kT_b \ , \tag{26}$$

or

$$T_b = \frac{c^2}{2k} \frac{I_\nu}{\nu^2} \ . \tag{27}$$

If the object we're observing really is a blackbody, then $T_b(\nu)$ is the actual temperature T of the object at all frequencies. More generally, however, the brightness temperature varies with frequency. Because the formula for T_b as a function of I_ν and ν is particularly simple in the Rayleigh-Jeans limit (low frequency), the brightness temperature is a quantity frequently used by radio and microwave astronomers, who deal with the low-frequency end of the electromagnetic spectrum.

The *effective temperature* $T_{\rm eff}$ is actually an alternative way of stating the energy flux from the surface of an object. That is, $T_{\rm eff}$ is the temperature for which the surface flux F, integrated over all frequencies, obeys the relation

$$F = \sigma T_{\rm eff}^4 \ , \tag{28}$$

or

$$T_{\rm eff} = (F/\sigma)^{1/4} \ . \tag{29}$$

Saying that the Sun's effective temperature is $T_{\rm eff} = 5778\,{\rm K}$ is equivalent to saying that the flux through the Sun's photosphere is $F_\odot = \sigma T_{\rm eff}^4 = 6.32 \times 10^{10}\,{\rm erg\,s^{-1}\,cm^{-2}}$ – that's 6.32 kilowatts per square centimeter. Here at the Earth's location, we are at a distance $r = 1\,{\rm AU} = 216\,{\rm R_\odot}$ from the Sun's center, so the flux measured by Earthlings is $F = F_\odot(R_\odot/r)^2 = 6.32 \times 10^{10}\,{\rm erg\,s^{-1}\,cm^{-2}}/(216)^2 = 1.36 \times 10^6\,{\rm erg\,s^{-1}\,cm^{-2}}$ – that's a mere 0.136 watts per square centimeter. This is the same flux you'd observe for a blackbody 1 AU in radius, and with a temperature of $T = 5778\,{\rm K}/(216)^{1/2} = 393\,{\rm K}$. However, what we observe from Earth is not an (approximate) blackbody spectrum with $T = 393\,\mathrm{K}$, but a dilution of an (approximate) blackbody spectrum with $T = 5778\,\mathrm{K}$. The effective temperature is a quantity frequently used by stellar astronomers, even when talking about T dwarfs like Gliese 229B (Figure 4), whose spectra are far from the blackbody ideal.

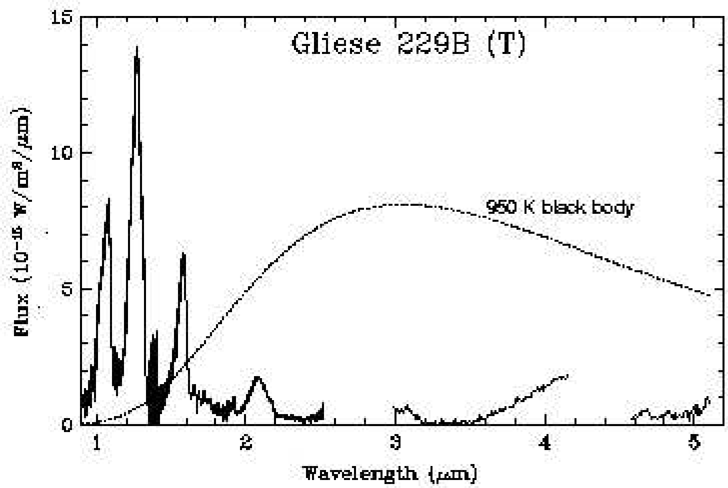

Figure 4: The flux F_λ of the T dwarf Gliese 229B, along with a blackbody spectrum of the same $T_{\rm eff}$ (Geballe et al. astro-ph/0102059).

The *color temperature* T_c of an object is actually an alternative way of stating the shape of its spectrum. There are different ways of defining the color temperature of an object. If the shape of the spectrum I_ν is close to that of a blackbody, the color temperature can be defined by finding the frequency $\nu_{\rm max}$ at which I_ν has a maximum, then estimating the color temperature from the Wien law:

$$T_c = \frac{\nu_{\rm max}}{5.9 \times 10^{10}\,\mathrm{Hz\,K^{-1}}} \ . \tag{30}$$

For multi-peak spectra like that of Gliese 229B, this definition of the color temperature is a fairly silly one. Another way of determining the color temperature is the measure the ratio of the specific intensities at two different frequencies, I_{ν_1}/I_{ν_2}, and find the temperature of a blackbody which has the same flux ratio. This is essentially what you do when you estimate a star's temperature from its $B - V$ color, for instance.[6] A warning: determining

[6]It's also what a blacksmith does when he estimates the temperature of a piece of hot iron from its color. For temperatures less than the melting point of iron (1811 K), $\nu_{\rm max}$

the color temperature from the flux ratio I_{ν_1}/I_{ν_2} is impossible when both frequencies lie in the Rayleigh-Jeans portion of the Planck function, since then

$$\frac{I_{\nu_1}}{I_{\nu_2}} = \frac{(2\nu_1^2/c^2)kT_c}{(2\nu_2^2/c^2)kT_c} = \frac{\nu_1^2}{\nu_2^2} \ , \tag{31}$$

independent of temperature. The B photometric band is centered at a frequency $\nu_1 = 6.8 \times 10^{14}\,\mathrm{Hz}$; the V band is at $\nu_2 = 5.5 \times 10^{14}\,\mathrm{Hz}$. Thus, at temperatures $T \gg h\nu_1/k = 33{,}000\,\mathrm{K}$, all blackbodies have the same $B - V$ color (which turns out to be $B - V = -0.46$). To estimate the temperatures of very hot O stars, you need to use higher frequency photometric bands.[7]

In summary: the brightness temperature T_b is a measure of the specific intensity I_ν at a given frequency; the effective temperature $T_{\rm eff}$ is a measure of the surface flux F; the color temperature is a measure of the shape of the specific intensity curve I_ν.

In discussing the radiative transfer equation, I treated the material through which the light passed as a mysterious black box. The specific intensity, I said, decreases by an amount $-\alpha_\nu I_\nu ds$ in traveling through a distance ds, due to something or other that's sucking up photons. (Conversely, the specific intensity increases by an amount $j_\nu ds$ due to something or other that's spewing out photons.) It is enlightening to look a little more closely at the physical processes by which the specific intensity I_ν is decreased while passing through a thin slab of material. Basically, two things can happen.

- A photon can be totally *absorbed* by the material, thus ceasing to exist. The photon energy goes to heat up the matter, which then re-radiates the energy in the form of thermal emission.

- A photon can be *scattered* by an encounter with a charged particle, such as an electron. In the low energy limit of Thomson scattering, the photon's energy is unchanged by scattering.

The effects of scattering are easiest to see when the scattering is *isotropic* – that is, when the scattered photons are emitted equally at all solid angles.[8]

lies in the infrared, where the smith can't observe it; he's basically comparing the flux at the red end of the visible spectrum to the flux at the violet end.

[7]The $U - B$ color is a useful estimator for the temperature of O stars.

[8]Real scattering processes are not usually isotropic, but since it makes the math much easier, I'll assume isotropy for the present.

Isotropic scattering that leaves photon energy unchanged is called *coherent scattering* in the text.

Coherent scattering is characterized by the parameter σ_ν, the *scattering coefficient*, which has units of cm^{-1} in the cgs system. The scattering coefficient is the probability that a photon will be scattered as it travels a distance of one centimeter. If the material is homogeneous, the mean free path between scatterings is $\ell = 1/\sigma_\nu$.

If a photon starts out at the origin in an infinite homogeneous medium with scattering coefficient σ_ν, it will undergo a random walk (sometimes called a "drunkard's walk") away from the origin.[9] At the time the photon undergoes its N^{th} scattering, it will be at a location

$$\vec{R} = \vec{r_1} + \vec{r_2} + \ldots + \vec{r_N} \ , \tag{32}$$

where $\vec{r_i}$ is the displacement of the photon between the $(i-1)^{\mathrm{th}}$ scattering and the i^{th} scattering. The mean square value of $\vec{R}$ will be

$$\begin{aligned} \langle R^2 \rangle &= \langle r_1^2 \rangle + \langle r_2^2 \rangle + \ldots \langle r_N^2 \rangle \\ &+ 2\langle \vec{r_1} \cdot \vec{r_2} \rangle + 2\langle \vec{r_1} \cdot \vec{r_3} \rangle + \ldots \end{aligned} \tag{33}$$

The cross terms (involving the dot product of one displacement vector with another) all vanish for isotropic scattering, since the directions of the different steps are uncorrelated with each other in that case. Thus, the mean square of the distance traveled after N steps is

$$\langle R^2 \rangle = \langle r_1^2 \rangle + \langle r_2^2 \rangle + \ldots \langle r_N^2 \rangle = N \langle r_1^2 \rangle \approx N\ell^2 \ . \tag{34}$$

Thus, the rms distance traveled by a photon that has scattered N times is

$$\langle R^2 \rangle^{1/2} \approx \sqrt{N}\ell \ . \tag{35}$$

where ℓ is the mean free path of the photon.

Suppose you are observing at MDM and a dense cloud bank settles on the mountain. Even at high noon, you can't see more than a few meters away; let's take $\ell = 100\,\mathrm{cm}$. If the fog extends upward for 1 kilometer ($R = 10^5\,\mathrm{cm}$) over head, the number of times a photon from the Sun must

[9]To see the origin of the term "drunkard's walk" consider an extremely intoxicated individual who takes steps of length ℓ, but who chooses a totally random direction each time he takes a step.

scatter before reaching you is $N \approx (R/\ell)^2 \approx 10^6$. It must travel a total distance of $\sim N\ell \approx 10^8\,\mathrm{cm}$ to get through the cloud layer. Thus, the photon must travel 100 kilometers to penetrate a layer 1 kilometer thick. Since the fog bank has an optical depth of $\tau \sim R/\ell \sim 1000$, the sunlight you detect is highly isotropized: it appears to come from all directions (except from the opaque earth under your feet).