1 Monday, October 31: Relativistic Charged Particles

As I was saying, before the midterm exam intervened, in an inertial frame of reference K there exists an electric field $\vec{E}$ and a magnetic field $\vec{B}$ at a spacetime location $(\vec{r}, t)$. Another inertial frame of reference K' is moving relative to the first at a constant velocity $\vec{v} = \vec{\beta}c$. In the K' frame of reference, the spacetime location $(\vec{r}, t)$ transforms to $(\vec{r}', t')$ by the Lorentz transformation.

In the K' frame of reference,

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \tag{1}$$

and

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \ , \tag{2}$$

where $\vec{E}_{\parallel}$ and $\vec{B}_{\parallel}$ are the components of the E and B fields parallel to the relative velocity vector $\vec{\beta}$. The transformations for the components perpendicular to β are more complicated, and thus more interesting.

$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B}) \tag{3}$$

and

$$\vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \vec{\beta} \times \vec{E}) \ , \tag{4}$$

where γ is the usually relativistic "gamma factor", $\gamma \equiv (1-\beta^2)^{-1/2}$. We can, of course, also the transformations from the primed frame K' to the unprimed frame K, by switching primed and unprimed quantities and changing the sign of $\vec{\beta}$:

$$\vec{E}_{\perp} = \gamma(\vec{E}'_{\perp} - \vec{\beta} \times \vec{B}') \tag{5}$$

and

$$\vec{B}_{\perp} = \gamma(\vec{B}'_{\perp} + \vec{\beta} \times \vec{E}') \ . \tag{6}$$

If there exists a single charged particle that moves with a constant velocity (as seen from an inertial frame of reference), transforming its electromagnetic field between one inertial frame and another is a fairly straightforward task, outlined in section 4.6 of the textbook. One interesting application is a charged particle which is moving past you (an inertial observer) on a straight line at a constant speed. The relative speed of you and the charged particle is $v = \beta c$. The distance of closest approach between you and the particle (the *impact parameter*) is b. Choose a coordinate system so that you, in the K frame of reference, are at $\vec{r} = b\hat{e}_y$, and the particle is moving along the x axis, with $\vec{r} = vt\hat{e}_x$. (Thus, I am choosing the moment $t = 0$ to be the moment of closest approach between you and the charged particle.) In the K' frame of reference, in which the particle is at rest, the $\vec{B}'$ field is zero, and the $\vec{E}'$ field is a simple inverse square relation:

$$\vec{E}' = \frac{q\vec{r}'}{(r')^3} \ . \tag{7}$$

In the K frame of reference, in which you are at rest, the $\vec{B}$ field is no longer zero and the *vecE* field is no longer isotropic around the particle. In the limit of highly relativistic motion ($\beta \approx 1$, $\gamma \gg 1$), the nonzero components of the electromagnetic field are

$$E_x \approx -\frac{q\gamma ct}{(\gamma^2 c^2 t^2 + b^2)^{3/2}} \tag{8}$$

$$E_y \approx \frac{q\gamma b}{(\gamma^2 c^2 t^2 + b^2)^{3/2}} \tag{9}$$

$$B_z \approx E_y \ . \tag{10}$$

The E field is maximized when $t = 0$, and the point charge is at its closest approach. At this instant,

$$E_y \approx B_z \approx \gamma \frac{q}{b^2} \ , \tag{11}$$

larger by a factor of γ than the value of E_y you would measure in the non-relativistic case. On the other hand, E_y only has a high value when $\gamma c|t| \ll b$, or

$$|t| \ll \frac{1}{\gamma}\frac{b}{c} \ . \tag{12}$$

In the more leisurely non-relativistic case, E_y has a high value when $v|t| \ll b$, or

$$|t| \ll \frac{b}{v} \sim \frac{1}{\beta}\frac{b}{c} \ . \tag{13}$$

In the highly relativistic case, $\gamma \gg 1$, and the encounter time is brief. In the non-relativistic case, $\beta \ll 1$, and the encounter time is long. A highly relativistic charged particle moving past you will cause a much higher electric force in your vicinity than a slowly moving particle would, but the high electric force will last for a much shorter period of time.

Suppose, though, that the charged particle q is not moving at a constant velocity $\vec{v}$ with respect to the observer, but has an arbitrary trajectory $\vec{r}_q(t)$ in the frame of reference K of an inertial observer. In general, a frame of reference attached to the moving charged particle is not inertial – the particle may be speeding up, or slowing down, or changing its direction of motion. However, at any instant t (as measured by the inertial observer), there exists an inertial frame of reference K' in which the particle is *instantaneously* at rest. That is, at the exact instant t, it's at rest in that frame. It won't remain at rest if it's being accelerated, but for shortly before and after the moment t, its motion in the K' frame will be non-relativistic, and we can use the classical formulas that we've derived earlier. (For example, we can apply the extremely useful Larmor formula in the K' frame.) In any other inertial frame K, moving at a velocity $-\vec{v}$ relative to K', we can compute the properties of the charged particle, and the radiation it emits, by doing the appropriate transformations.

Let P' be the power radiated by the charged particle, as measured in its instantaneous rest frame K'. Let P be the power radiated, as measured in the other inertial frame K. As long as the radiation emitted has no net momentum in the K' frame,[1] it can be shown that

$$P = P' \ . \tag{14}$$

That is, if a charged particle radiates with a power P', as measured in an inertial frame where it is non-relativistic, it will radiate with the same power $P = P'$, in an inertial frame where it is highly relativistic. (However, the angular distribution of the power radiated may be very different in the two frames.)

In the K' frame, we can compute the power radiated by using the non-relativistic formula for an accelerated point charge:

$$P' = \frac{2q^2}{3c^3}|\vec{a}'|^2 \ , \tag{15}$$

where $\vec{a}'$ is the acceleration of the particle as measured in the K' frame. In

[1]The symmetric radiation produced by a non-relativistic accelerated particle satisfies the 'no net momentum' criterion.

making the transformation to the K frame, it is useful to write

$$P' = \frac{2q^2}{3c^3}\left[(a'_\parallel)^2 + (a'_\perp)^2\right] \ , \tag{16}$$

since (as you can probably guess), the component of $\vec{a}'$ parallel to the $\vec{v}$ vector, which we call $a'_\parallel$, transforms differently from the component of $\vec{a}'$ perpendicular to $\vec{v}$; we call this component $a'_\perp$. Rybicki and Lightman state that

$$a'_\parallel = \gamma^3 a_\parallel \tag{17}$$

and

$$a'_\perp = \gamma^2 a_\perp \ , \tag{18}$$

but leave the proof of their statement to problem 4.3 (a classic "exercise left for the reader"). Note that the acceleration measured in the instantaneous rest frame K' is always larger than the acceleration measured in an inertial frame K that's in motion relative to K'. With the transformation of accelerations between frames, we may write

$$P = P' = \frac{2q^2}{3c^3}\left[(a'_\parallel)^2 + (a'_\perp)^2\right] \tag{19}$$

$$= \frac{2q^2\gamma^4}{3c^3}(\gamma^2 a_\parallel^2 + a_\perp^2) \ . \tag{20}$$

Doing the transformation of the angular power distribution is a bit more complicated. Remember, in the instantaneous rest frame K', we can use the non-relativistic formula

$$\frac{dP'}{d\Omega'} = \frac{q^2(a')^2}{4\pi c^3}\sin^2\Theta' \ , \tag{21}$$

where Θ' is the angle between $\vec{a}'$ and the direction of radiation, as measured in the K' frame. But what will the angular power distribution $dP/d\Omega$ be in an inertial frame K, moving at a velocity $-\vec{v}$ relative to the particle's instantaneous rest frame? Doing a partial transformation is relatively easy. Referring to Rybicki and Lightman, we find

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{(\gamma^2 a_\parallel^2 + a_\perp^2)}{(1-\beta\cos\theta)^4}\sin^2\Theta' \ . \tag{22}$$

It's converting from the emission angle Θ' as measured in the K' frame to that measured in the K frame that is difficult. Even in this imperfect state of conversion, we can notice some interesting results for the radiation from an accelerated particle. Since $\beta = (1 - 1/\gamma^2)^{1/2}$, for highly relativistic particles, $\beta \approx 1 - 1/(2\gamma^2)$ is very close to one. That means that the term

$$\frac{1}{(1 - \beta \cos\theta)^4} \qquad (23)$$

is strongly peaked near $\cos\theta \approx 1$ for relativistic particles. Thus, we expect to see strong beaming of the radiation in the direction $\theta \approx 0$; that's the direction of motion of the particle as seen by the observer in the K frame of reference. The exact formula for $dP/d\Omega$ is written in Rybicki and Lightman for the cases in which the acceleration $\vec{a}$ is parallel to the $\vec{v}$ vector, and in which the acceleration is perpendicular to $\vec{v}$. When the acceleration is perpendicular to the relative velocity of particle and observer (Figure 1), the majority of

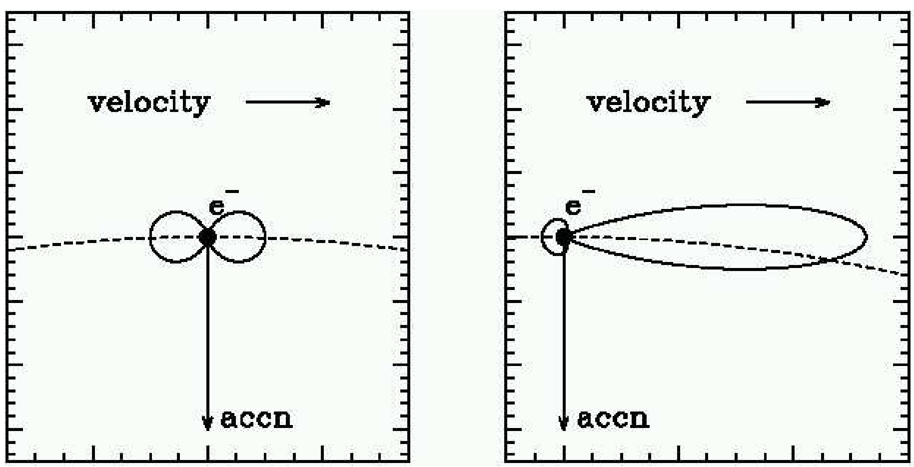

Figure 1: Radiated power in the instantaneous rest frame (left) and the observer's rest frame (right).

the power radiated is in a single lobe of opening angle $\sim 1/\gamma$, pointing in the direction of the particle's motion (as determined by the K frame observer).

2 Wednesday, November 2: Bremsstrahlung for Beginners

We now have mastered the art of computing the radiation from a single accelerated charged particle, even when the charged particle is moving at very high speeds ($\gamma \gg 1$) relative to the observer. As astronomers, however, we typically have to deal with the light emitted by many, many charged particles simultaneously. To see how the light from individual charged particles adds together to form the spectrum of light that we actually observe, let's start by considering the case of *thermal bremsstrahlung*. As I noted near the beginning of the course, bremsstrahlung is the term for radiation that is produced when one charged particle is accelerated by its interaction with another charged particle. Thermal bremsstrahlung is produced when the relative motion of charged particles is due to the fact that they're in a hot gas, and thus have random thermal velocities.

Consider a hot ionized gas, which contains a number density n_e of free electrons, each with charge $-e = -4.8 \times 10^{-10}$ esu, and a number density n_i of positively charged ions, each with charge $+Ze$. I'll make the simplifying assumption that all the ions have the same charge Z; the generalization to ions with different charges is left as an exercise to the reader. Today, I'll focus on non-relativistic bremsstrahlung, which occurs when the temperature T of the gas is $T \ll m_e c^2/k \sim 6 \times 10^9$ K. I'll begin by considering a single electron-ion interaction, then I'll generalize to find the expected power spectrum of radiation from a large expanse of hot ionized gas.

As an aside, we don't expect significant radiation from the interaction of two particles with identical mass and identical charge (two electrons, for instance, or two identical positive ions). In the case of particles with the same mass, the center of mass is always midway between the particles; if the charge of the particles is the same, the electric dipole moment then vanishes. Therefore, there will be no electric dipole radiation from electron-electron interactions, or from the interaction of identical ions. The source of bremsstrahlung from a hot gas is the interactions between electrons and ions; since it's the low-mass electrons that have the greater acceleration, the electrons are the predominant source of radiation (Figure 2). Let's start simply, with a system that consists of one electron and one positive ion. The electron and ion are initially a large distance apart, and their initial relative velocity is $\vec{v}$, where $v \ll c$. The impact parameter of the electron is b; that is, if the electron continued to move on a straight line relative to the ion (ignoring the electromagnetic forces), its closest distance to the ion would be b. Since this is a non-relativistic system, we can approximate the force between electron and ion as being an inverse square electrostatic force:

$$F = -\frac{Ze^2}{R^2} \,, \tag{24}$$

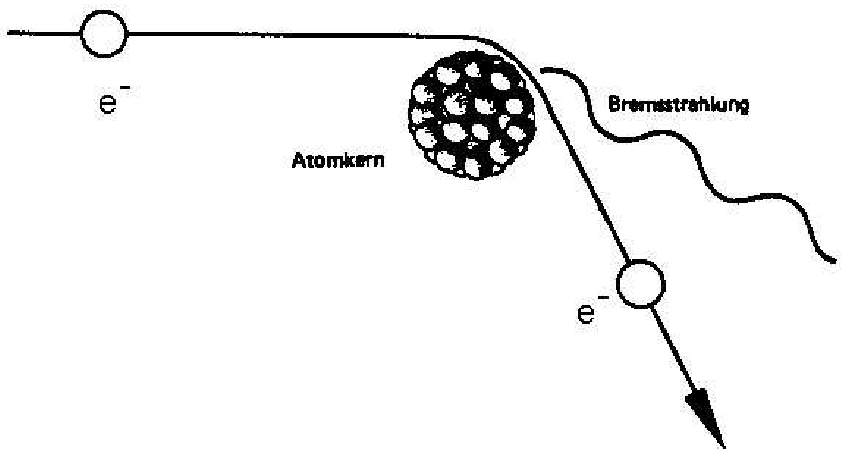

Figure 2: Bremsstrahlung: cartoon version.

where R is the distance between electron and ion. Isaac Newton would tell us that the orbit of the electron relative to the ion will be a hyperbola if the system is unbound.

By adopting the *impulse approximation*, we can make a rapid order-of-magnitude estimate of the radiation emitted by the electron. Suppose that we set $t = 0$ to be the time of closest approach. The impulse approximation states that the force on the electron is zero until a time $-t_{\rm col}/2 \sim -b/v$.[2] The electron then feels a force of order $F \sim -Ze^2/b^2$, in in the direction perpendicular to its initial velocity, until a time $+t_{\rm col}/2 \sim b/v$. The net change of the electron's velocity is then perpendicular to its initial velocity, and has a magnitude

$$\Delta v \sim \frac{F}{m_e}(2\Delta t) \sim \frac{Ze^2}{m_e b^2}\frac{2b}{v} \sim \frac{2Ze^2}{m_e b v} \ . \tag{25}$$

The name "impulse approximation" comes from the fact that the electron is assumed to acquire its Δv in one brief impulse; a single short, sharp, shock administered by the positive ion. The impulse approximation is only a good one when the change in velocity, Δv, is small compared to the initial velocity, v. This criterion is met when

$$\frac{2Ze^2}{m_e b v} < v \ , \tag{26}$$

[2]The characteristic timescale for the collision, $t_{\rm col} \sim 2b/v$, is known as the *collision time*.

or

$$b > b_{\rm min} \equiv \frac{2Ze^2}{m_e v^2} = 2Zr_0 \left(\frac{v}{c}\right)^{-2} \ . \tag{27}$$

If the impulse approximation holds, then for a time $t \sim t_{\rm col} \sim 2b/v$, the electron undergoes an acceleration

$$a \sim \frac{F}{m_e} \sim \frac{Ze^2}{m_e b^2} \ , \tag{28}$$

and thus radiates energy at the rate (from the Larmor formula)

$$P = \frac{2e^2 a^2}{3c^3} \sim \frac{2e^2}{3c^3} \frac{Z^2 e^4}{m_e^2 b^4} \sim \frac{2Z^2 e^6}{3m_e^2 c^3 b^4} \ . \tag{29}$$

The total energy radiated during the electron's encounter with the ion is thus

$$W = P \cdot t_{\rm col} \sim \frac{2Z^2 e^6}{3m_e^2 c^3 b^4} \frac{2b}{v} \sim \frac{4Z^2 e^6}{3m_e^2 c^3 b^3 v} \ . \tag{30}$$

The distribution in angular frequency ω of the emitted radiation is a useful function to calculate. The duration of the electron – ion interaction, $t_{\rm col} \sim 2b/v$, corresponds to an angular frequency

$$\omega_{\rm col} \sim \frac{2\pi}{t_{\rm col}} \sim \frac{\pi v}{b} \ . \tag{31}$$

For low angular frequencies, $\omega \ll \omega_{\rm col}$, the acceleration of the electron looks like a delta function; that is, the duration of the collision is much shorter than the period of a low-frequency wave. A delta function in time, when you perform a Fourier transform, corresponds to a uniform function in frequency. Thus, we expect $dW/d\omega$, the distribution of emitted energy as a function of frequency, to be roughly uniform up to a maximum frequency $\sim \omega_{\rm col}$. The spectral distribution of the radiated energy is then

$$\frac{dW}{d\omega} \sim \frac{W}{\omega_{\rm max}} \sim \frac{b}{\pi v} W \sim \frac{4Z^2 e^6}{3\pi m_e^2 c^3 b^2 v^2} \ , \tag{32}$$

when $\omega \ll \omega_{\rm max}$ and $dW/d\omega = 0$ when $\omega \gg \omega_{\rm max}$.

I find, when I consult the textbook, that the result in equation (32) is smaller than a factor of two than the more painstakingly derived result of

Rybicki and Lightman. I bow to their expertise, and will henceforth assume that

$$\frac{dW}{d\omega} \approx \frac{8Z^2e^6}{3\pi m_e^2 c^3 b^2 v^2} \ . \tag{33}$$

This result still represents the radiation emitted by a single electron during a single encounter with an ion. We still need to compute the spectrum of light expected from a large ensemble of electrons and ions. Suppose that the number density of free electrons in a medium is n_e and the number density of ions (each of which has electric charge Ze) is n_i. The relative velocity of the electrons and ions is assumed to have the value v. The flux of electrons passing by any particular ion will have the value vn_e (electrons per unit time per unit area). At a distance b from the ion, the area element is $dA = 2\pi b db$, so the number of electrons per unit time passing the ion at a distance b will be

$$\frac{dN}{dtdb}db = vn_e dA = vn_e 2\pi b db \ . \tag{34}$$

The power radiated by those electrons will be the rate at which electrons pass times the energy per electron:

$$\frac{dW}{d\omega dtdb} = \frac{dN}{dtdb}\frac{dW}{d\omega} = n_e 2\pi b v \frac{8Z^2e^6}{3\pi m_e^2 c^3 b^2 v^2} \approx n_e \frac{16Z^2e^6}{3m_e^2 c^3 bv} \ . \tag{35}$$

To compute the power per unit volume radiated by all the electrons in the medium (also known as the *specific intensity* of the medium), we integrate the power per ion over all possible impact parameters b, then multiply by the number density of ions:

$$\frac{dW}{d\omega dtdV} = n_i n_e \frac{16Z^2e^6}{3m_e^2 c^3 v}\int_{b_{\rm min}}^{b_{\rm max}} \frac{db}{b} \tag{36}$$

Note that we need both an upper and lower cutoff on the integral; otherwise the power will be logarithmically divergent. As it is, we have a formula that states

$$\frac{dW}{d\omega dtdV} \approx n_i n_e \frac{16Z^2e^6}{3m_e^2 c^3 v} \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \ . \tag{37}$$

We don't need to know the values of $b_{\rm max}$ and $b_{\rm min}$ with extreme exactness, since they only enter our formula logarithmically, but we should know what they are to within an order of magnitude. We can set $b_{\rm max}$ by noting that for a fixed value of b, no power is radiated for frequencies higher than $\omega_{\rm col} \sim$

$\pi v/b$. Thus, for a fixed value of ω, no power is contributed by encounters with impact parameters greater than $b_{\rm max} \sim \pi v/\omega$. In the classical regime, we can take for the lower limit on b, the value $b_{\rm min}$ for which the impulse approximation breaks down:

$$b_{\rm min} \sim \frac{2Ze^2}{m_e v^2} \ . \tag{38}$$

For high electron energies, the cutoff is imposed by quantum mechanical factors.

Generally, physicists write

$$\frac{dW}{d\omega dV dt} = Z^2 n_i n_e \frac{16\pi e^6}{3\sqrt{3}c^3 m_e^2 v} g_{ff} \ , \tag{39}$$

where g_{ff} is the *Gaunt factor*, which in general depends on both the electron speed v and the radiation frequency ω.[3] By comparison with equation (37), we find that the Gaunt factor is equivalent to

$$g_{ff}(v,\omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \ . \tag{40}$$

There exist tabulations of g_{ff} in different temperature and frequency regimes. (However, it is generally a factor of order unity.)

3 Friday, November 4: Advanced Bremsstrahlung

Suppose that you have a gas containing a number density n_e of free electrons (each with charge $-e$) and a number density n_i of positively charged ions (each with charge $+Ze$). If the electrons are moving relative to the ions with a typical velocity v then the gas will emit bremsstrahlung radiation with a specific intensity

$$\frac{dW}{d\nu dV dt} \propto v^{-1} Z^2 n_e n_i g_{ff}(\nu, v) \ , \tag{41}$$

where g_{ff} is the Gaunt factor.[4] In an x-ray tube (Figure 3), the relative

[3]The subscript "ff" refers to the fact that bremsstrahlung emission is also called "free-free" emission; the electron starts free and ends free.

[4]I have changed from using the angular frequency ω of the radiation to the frequency $\nu = \omega/(2\pi)$. Although ω and ν are interchangeable, ν is more frequently used by astronomers.

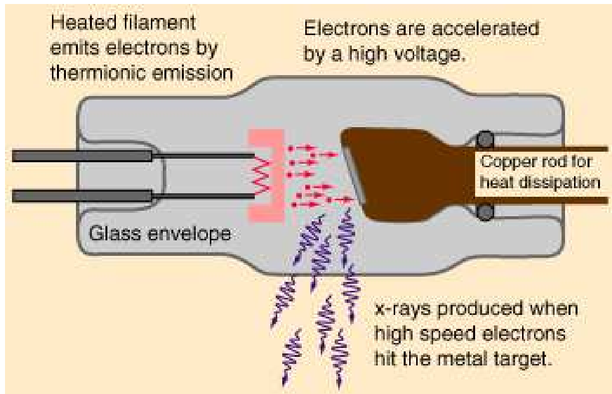

Figure 3: Production of x-rays by bremsstrahlung.

velocity of electrons and atomic nuclei is produced by a high voltage vacuum tube (about 100 kilovolts is typical). The rapidly moving electrons penetrate a metallic target. If an electron comes close to an atomic nucleus (well inside the electron cloud surrounding it), it is accelerated by its electromagnetic interaction with the nucleus, and emits bremsstrahlung.

If the relative motion of the electrons and ions is due to *thermal* motions, we expect that $v^2 \sim kT/m_e$, and thus

$$\frac{dW}{d\nu dV dt} \propto T^{-1/2} Z^2 n_e n_i g_{ff}(\nu, T) \ . \tag{42}$$

More precisely, an integration over the Maxwell distribution of electron velocities at a given temperature yields

$$\frac{dW}{d\nu dV dt} = \frac{2^5 \pi e^6}{3 m_e c^3} \left(\frac{2\pi}{3 m_e k T} \right)^{1/2} Z^2 n_e n_i \bar{g}_{ff} e^{-h\nu/kT} \ , \tag{43}$$

where $\bar{g}_{ff}(\nu, T)$ is the Gaunt factor averaged over all electron velocities at a fixed temperature T. The exponential cutoff in equation (43) is a simple quantum effect, of the sort Max Planck would appreciate. Light of wavelength ν comes in quanta of energy $e = h\nu$. An electron can't produce a bremsstrahlung photon with energy greater than its initial kinetic energy. At a temperature T, there is an exponentially small number of electrons with kinetic energy $> kT$; thus, there can only be an exponentially small number of photons produced with $h\nu > kT$. Note, in equation (43), that for frequencies $h\nu \ll kT$, the only dependence on frequency is through the

Gaunt factor $\bar{g}_{ff}$. At low frequencies, the Gaunt factor depends only logarithmically on ν; thus, bremsstrahlung spectra tend to be nearly flat up to the exponential cutoff at $h\nu \sim kT$.[5]

Integrated over frequency, the power per unit volume from non-relativistic thermal bremsstrahlung is

$$\frac{dW}{dtdV} = \frac{2^5 \pi e^6}{3hm_e c^3} \left(\frac{2\pi kT}{3m_e} \right)^{1/2} Z^2 n_e n_i \bar{g}_B \ , \tag{44}$$

where $\bar{g}_B(T)$ is the frequency-averaged value of $\bar{g}_{ff}(\nu, T)$. Thankfully, over a wide range of temperatures, the approximation $\bar{g}_B \approx 1.2$ is within 20% of the truth – good enough for astronomical purposes. If all quantities are in cgs units, this yields

$$\frac{dW}{dtdV} \approx 1.7 \times 10^{-27} \, \mathrm{erg\, s^{-1}\, cm^{-3}} T^{1/2} Z^2 n_e n_i \ . \tag{45}$$

In the hot coronal bubbles that fill the interstellar medium, the typical temperature is $T \sim 10^6\,\mathrm{K}$ and $n_e \sim n_i \sim 10^{-2}\,\mathrm{cm^{-3}}$. If we assume the interstellar gas is mainly hydrogen ($Z = 1$), the specific intensity from bremssstrahlung will be $dW/dtdV \sim 2 \times 10^{-28}\,\mathrm{erg\,s^{-1}\,cm^{-3}}$; this means that a cube of coronal gas 1 AU on a side would produce less than 100 kilowatts of bremsstrahlung power. The Local Bubble, in which the Sun is located, has a radius of $R \sim 50\,\mathrm{pc} \sim 1.5 \times 10^{20}\,\mathrm{cm}$, and a volume, assuming spherical symmetry, of $V \sim (4\pi/3)R^3 \sim 1.5 \times 10^{61}\,\mathrm{cm^3}$. Thus, its total bremsstrahlung luminosity will be

$$L = \frac{dW}{dtdV} V \sim (2 \times 10^{-28}\,\mathrm{erg\,s^{-1}\,cm^{-3}})(1.5 \times 10^{61}\,\mathrm{cm^3}) \sim 3 \times 10^{33}\,\mathrm{erg\,s^{-1}} \ , \tag{46}$$

or roughly one solar luminosity.[6] The maximum frequency at which the bremsstrahlung will be radiated is $\nu \sim kT/h \sim 2 \times 10^{16}\,\mathrm{Hz}$, in the ultraviolet.

The bremsstrahlung process can also be run in reverse. That is, if an electron passing close to a positive ion can emit a photon, then an electron

[5] For comparison, the spectrum of a blackbody has a fairly steep $I \propto \nu^2$ dependence until you reach its exponential cutoff at $h\nu \sim kT$.

[6] The total mass of gas in the Local Bubble is about $120\,\mathrm{M_\odot}$, so the mass-to-light ratio is high.

passing close to a positive ion can also *absorb* a photon.[7] Since bremsstrahlung is also referred to as "free – free emission", the inverse process is known as "free – free absorption". Let's consider a large cloud of ionized gas which both emits light by free – free emission and absorbs light by free – free absorption. If we rummage through our memories of radiative transfer, we remember that the *emission coefficient* j_ν is the energy emitted per unit time per unit volume per unit solid angle. For isotropic bremsstrahlung emission,

$$j_\nu = \frac{1}{4\pi}\frac{dW}{d\nu dt dV} \ , \tag{47}$$

where the value of $dW/d\nu dt dV$ is given by equation (43). If the gas cloud is in thermal equilibrium, the rate of emission, j_ν, is equal to the rate of absorption, $\alpha_\nu^{ff} B_\nu(T)$, where α_ν^{ff} is the absorption coefficient for free – free absorption and $B_\nu(T)$ is the Planck function. Thus, the relation

$$j_\nu = \alpha_\nu^{ff} B_\nu(T) \tag{48}$$

implies

$$\alpha_\nu^{ff} = \frac{1}{4\pi}\frac{dW}{d\nu dt dV}\frac{1}{B_\nu(T)} \ , \tag{49}$$

or

$$\alpha_\nu^{ff} = \frac{8e^6}{3m_e c^3}\left(\frac{2\pi}{3m_e kT}\right)^{1/2} Z^2 n_e n_i \bar{g}_{ff} e^{-h\nu/kT} \cdot \frac{c^2}{2h\nu^3}(e^{h\nu/kT} - 1) \tag{50}$$

$$= \frac{4e^6}{hm_e c}\left(\frac{2\pi}{3m_e kT}\right)^{1/2} \frac{Z^2 n_e n_i}{\nu^3}\bar{g}_{ff}(1 - e^{-h\nu/kT}) \ . \tag{51}$$

This absorption coefficient is much more complicated than the Thomson scattering coefficient ($\sigma_\nu = \sigma_T n_e$), but it's useful to know, since free – free absorption is the dominant opacity source in some temperature and frequency regimes.

In the high-frequency limit ($h\nu \gg kT$), we are on the Wien tail of the Planck function, and $\alpha_\nu^{ff} \propto \nu^{-3}$. In the low-frequency limit ($h\nu \ll kT$), we are in the Rayleigh-Taylor portion of the Planck function, and $\alpha_\nu^{ff} \propto \nu^{-2}$. Because of the steep decline of the absorption coefficient with increasing

[7]For an electron to absorb a photon, there must be a third body present – such as the ion – to carry away some of the momentum. Otherwise, it would be impossible to conserve both momentum and energy.

frequency, free – free absorption tends to be an important physical process at low frequencies. In the low-frequency limit, the numerical value of the absorption coefficient is

$$\alpha_\nu^{ff} = 0.018\,\mathrm{cm}^{-1} T^{-3/2} \frac{Z^2 n_e n_i}{\nu^2} \bar{g}_{ff} \ , \tag{52}$$

when all quantities are in cgs units. In the Local Bubble, where $T \sim 10^6$K, and $n_e \sim n_i \sim 10^{-2}\,\mathrm{cm}^{-3}$, the free – free absorption coefficient is

$$\alpha_\nu^{ff} \sim \frac{2 \times 10^{-15}\,\mathrm{cm}^{-1}}{\nu^2} \ . \tag{53}$$

If we are at a distance $R \sim 50\,\mathrm{pc} \sim 1.5 \times 10^{20}$ cm from the edge of the Local Bubble, then the Bubble will be optically thick when

$$\alpha_\nu^{ff} R \sim \frac{3 \times 10^5}{\nu^2} > 1 \ . \tag{54}$$

This corresponds to frequencies $\nu < 500\,\mathrm{Hz}$, or wavelengths $\lambda > 600\,\mathrm{km}$. One reason why astronomers don't observe at extremely low frequencies is that the local interstellar medium is opaque in the ELF bandpass (usually defined as $3\,\mathrm{Hz} < \nu < 300\,\mathrm{Hz}$.)[8]

[8] Another reason is that powerful ELF transmitters are used to send messages to deeply submerged submarines. Thus, any signal you picked up would be likely to be manmade.