1 Wednesday, November 16: Return of Synchrotron

Both bremsstrahlung and synchrotron radiation can be emitted by hot ionized gas. How can you tell whether the light you observe from a distant galaxy is bremsstrahlung or synchrotron? One way of distinguishing, as we have seen, is by the spectrum of light emitted. Bremsstrahlung typically has a flux F_ν that is nearly constant with frequency between the free-free absorption cutoff at low frequencies and the exponential Planck cutoff at high frequencies. Synchrotron emission has a steeper dependence on frequency: $F_\nu \propto \nu^{-s}$, with $s \approx 0.7$ for typical synchrotron emission from interstellar gas. If the distribution of relativistic electron energies is $n(\epsilon) \propto \epsilon^{-p}$, the relation between p and s is

$$s = \frac{p-1}{2} . \tag{1}$$

Thus, $s \approx 0.7$ implies $p \approx 2.4$.

Another way of determining whether the source of light is bremsstrahlung or synchrotron radiation is to look at its *polarization*. Thermal bremsstrahlung is unpolarized. Since the thermal motions of the free electrons and ions are random, there are no preferred axes in the problem. Thus, although the light from an individual electron – ion encounter is polarized, the light from many, many such encounters added together has no net polarization. However, synchrotron radiation does have a preferred axis – the direction of the magnetic flux density $\vec{B}$. The calculation of the polarization of synchrotron radiation is a bit tedious, involving more modified Bessel functions, so I'll let you work through that bit of the textbook at your leisure. It turns out that the synchrotron radiation has a net *linear* polarization. The axis of polarization is perpendicular to the magnetic field $\vec{B}$ as projected onto the plane of the sky. The direction of polarization, as shown in Figure 1 for the galaxy M51, indicates the (projected) direction of the magnetic field. For electrons with $n(\epsilon) \propto \epsilon^{-p}$, the degree of linear polarization, integrated over all frequencies and all electron energies, is

$$\Pi = \frac{p+1}{p+7/3} . \tag{2}$$

If $p \approx 2.4$, the degree of polarization is $\Pi \approx 0.72$. A 72 percent polarization is quite high; remember that the polarization of starlight scattered from

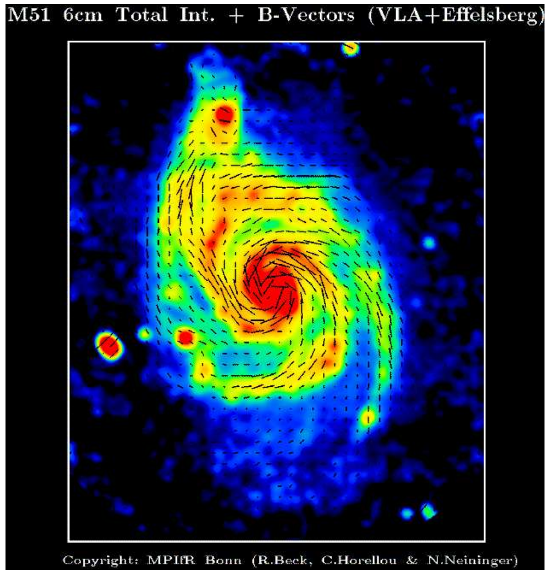

Figure 1: Synchrotron intensity at $\nu = 4.86\,\mathrm{GHz}$ (false color) and magnetic field orientation (straight lines) in M51; length of line is proportional to degree of polarization.

dust was only a few percent. Synchrotron emission thus provides a powerful tool for studying the magnetic fields within galaxies. The linear polarization shows the projected direction of $\vec{B}$. Since the synchrotron power is (as we saw last week)

$$\frac{dP}{dV d\nu} \propto n_e B^{(p+1)/2} \nu^{-(p-1)/2} \,, \tag{3}$$

the intensity of synchrotron emission at a given ν tells you about the amplitude of $\vec{B}$.

The synchrotron emission is particularly strong in *radio galaxies* such as Cygnus A (Figure 2). In a radio galaxy, synchrotron emission is the main source of electromagnetic radiation at frequencies lower than $\sim 300\,\mathrm{GHz}$, corresponding to wavelengths longer than $\sim 1\,\mathrm{mm}$. In an extremely luminous radio galaxy such as Cygnus A, the synchrotron radiation comes primarily from two large radio lobes, each much larger than the central stellar distribution. Cygnus A is at a distance $d \approx 240\,\mathrm{Mpc}$. At this distance, an angle of $1\,\mathrm{arcsec}$ corresponds to a linear distance of 240 million AU, or $1.2\,\mathrm{kpc}$. The radio lobes of Cygnus A, which are about 40 arcseconds across, must each have a diameter of $D \approx 50\,\mathrm{kpc}$. Each of the lobes has a radio luminosity of

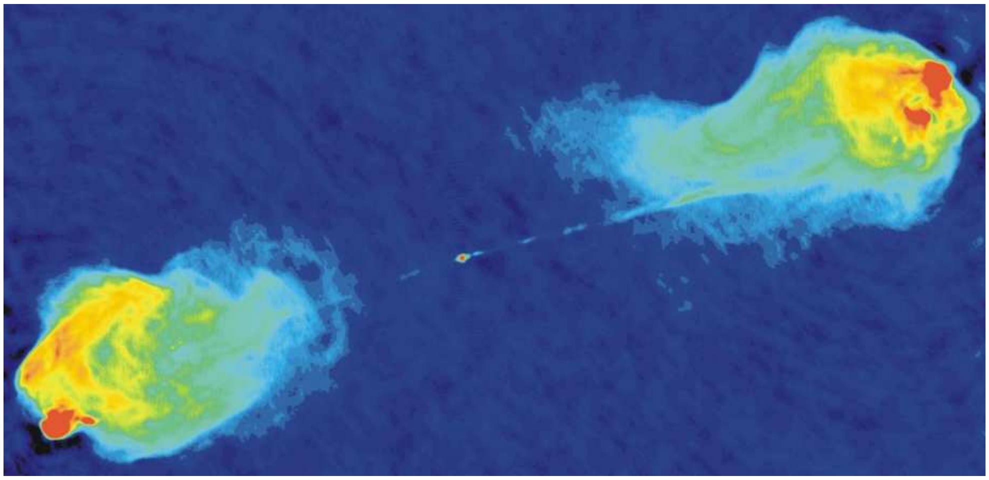

Figure 2: False color image of Cygnus A, at a frequency of 5.0 GHz (field of view is 2 arcmin across).

$P \approx 10^{45}\,\mathrm{erg\,s^{-1}}$.

If the typical magnetic flux density in the lobes is B, then the magnetic energy density is

$$U_B = \frac{B^2}{8\pi} \approx 4 \times 10^{-10}\,\mathrm{erg\,cm^{-3}} \left(\frac{B}{10^{-4}\,\mathrm{gauss}} \right)^2 , \tag{4}$$

scaling to a typical radio lobe magnetic field strength. The total magnetic energy in one of the radio lobes is then

$$E_B = \frac{B^2}{8\pi} \left(\frac{\pi}{6} D^3 \right) \approx 8 \times 10^{59}\,\mathrm{erg} \left(\frac{B}{10^{-4}\,\mathrm{gauss}} \right)^2 . \tag{5}$$

This is a pretty large amount of energy; roughly equal to one billion times the energy radiated by the Sun during its entire main sequence lifetime.

Suppose that the lobes of Cygnus A contain relativistic electrons with $n(\gamma) \propto \gamma^{-p}$ between a minimum Lorentz factor γ_1 and a maximum Lorentz factor γ_2. If $p > 1$ and $\gamma_2 \gg \gamma_1$, then we can make the normalization

$$n(\gamma) = (p-1) \frac{n_e}{\gamma_1} \left(\frac{\gamma}{\gamma_1} \right)^{-p} , \tag{6}$$

where n_e is the total number density of relativistic electrons. The energy density of the electrons is then

$$U_e = \int_{\gamma_1}^{\gamma_2} (\gamma m_e c^2) n(\gamma) d\gamma \tag{7}$$

$$= (p-1) n_e (\gamma_1 m_e c^2) \int_1^{\gamma_2/\gamma_1} x^{-p+1} dx \ , \tag{8}$$

where I've made the change of variables $x \equiv \gamma/\gamma_1$. If $p > 2$, as we expect for real distributions of electrons, the integral converges, and for $\gamma_2 \gg \gamma_1$,

$$U_e \approx \frac{p-1}{p-2} n_e (\gamma_1 m_e c^2) \tag{9}$$

$$\approx 3 \times 10^{-10} \, \mathrm{erg\,cm^{-3}} \left(\frac{n_e}{10^{-5}\,\mathrm{cm^{-3}}} \right) \left(\frac{\gamma_1}{10} \right) \ . \tag{10}$$

Note that I have ever-so-casually chosen values for n_e, γ_1, and B that have made the energy density of the magnetic field (equation 4) nearly equal to the energy density of the electrons (equation 10).[1] It has not escaped the notice of astronomers that plausible choices for n_e, γ_1, and B provide comparable energy densities for the magnetic field and for the relativistic electrons. This "equipartition" of magnetic field energy and electron energy is a subject of speculation among radio astronomers. Do the magnetic field and electrons somehow swap energy back and forth to attain equipartition? If so, relativistic magnetohydrodynamic turbulence is probably involved. If you want to study relativistic magnetohydrodynamic turbulence, I encourage you to do so. I'll be cheering you on from the sidelines.

The power per unit volume per unit frequency, $dP/dV d\nu \propto \nu^{-(p+1)/2}$, has a high-frequency cutoff at the highest frequency produced by the highest-energy electrons in the emitting region ($\nu_{\rm max} \sim \gamma_2^2 \nu_{\rm cyc}$). There is also a low-frequency cutoff. As you might recall, a *bremsstrahlung* spectrum is cut off at the low-frequency end by free-free absorption, which is the inverse process to bremsstrahlung (otherwise known as "free-free emission"). You might expect, then, that a synchrotron spectrum would be cut off at the low frequency end by *synchrotron self-absorption*, which is the inverse process to synchrotron emission. If you expected this, your expectation would not be disappointed.

Unfortunately, computing the synchrotron absorption coefficient, $\alpha_\nu^{\rm syn}$, is more difficult than computing the free-free absorption coefficient, α_ν^{ff}. In

[1] I have also assumed $p \approx 2.4$.

part, this is because the synchrotron emission is not necessarily isotropic, since $\vec{B}$ imposes a preferred direction. We can wiggle our way out of this difficulty by declaring that we will look at a region in which the magnetic field $\vec{B}$ is badly tangled, so that it has no net direction. With that assumption, we can set the emission coefficient j_ν equal to the angle-averaged synchrotron power per unit volume per unit frequency:

$$j_\nu = \frac{1}{4\pi} \frac{dP}{dV d\nu} \propto n_e B^{(p+1)/2} \nu^{-(p-1)/2} \ . \tag{11}$$

A more difficult problem is posed by the fact that synchrotron emission is *non-thermal*, so that we cannot assume that the source function S_ν is equal to the Planck function B_ν. Without that handy assumption, the synchrotron absorption coefficient must be calculated by the tedious use of Einstein coefficients. For a power law distribution of electron energies, $n(\gamma) \propto \gamma^{-p}$, the synchrotron absorption coefficient has the dependence

$$\alpha_\nu^{\rm syn} \propto n_e B^{(p+2)/2} \nu^{-(p+4)/2} \ . \tag{12}$$

When $p \approx 2.4$, the frequency dependence of the synchrotron absorption coefficient is $\alpha_\nu \propto \nu^{-3.2}$. This steep dependence on frequency means that a region containing a magnetic field and relativistic electrons will be optically thick to synchrotron radiation at sufficiently low frequencies.

In equilibrium, the emission coefficient j_ν is equal to the absorption coefficient $\alpha_\nu^{\rm syn}$ times the source function S_ν. Thus, the synchrotron source function is

$$S_\nu = \frac{j_\nu}{\alpha_\nu^{\rm syn}} \propto B^{-1/2} \nu^{5/2} \ . \tag{13}$$

Note that the source function is independent of γ, the power-law index for the relativistic electrons. At low frequencies, the synchrotron-emitting region will be opaque, and the observed intensity I_ν will be proportional to the source function: $I_\nu \propto S_\nu \propto \nu^{5/2}$. At high frequencies, the region will be transparent, and the observed intensity will be proportional to the emission coefficient: $I_\nu \propto j_\nu \propto \nu^{-(p-1)/2} \propto \nu^{-0.7}$. The frequency at which the synchrotron-emitting region makes the transition from transparent to opaque depends on the depth of the region, its number density of relativistic electrons, and its magnetic flux density.

2 Friday, November 18: Compton Scattering

About a month ago (on October 19 and 21, to be precise), we investigated the scattering of light by photons in the limit that $h\nu \ll m_e c^2$. In that limit, it is useful to regard light as an electromagnetic wave. The varying electric field causes the electron to oscillate, and the oscillating dipole causes a re-emission of light from the electron. As you will recall (I hope), this classical scattering of waves, called Thomson scattering, leaves the frequency of light unchanged, and has a cross-section $\sigma_T = (8\pi/3)r_0^2 = 6.65 \times 10^{-25}\,\text{cm}^2$, where r_0 is the classical electron radius.

But what about the limit of high photon energy, with $h\nu \gg m_e c^2$? Although most photons in the universe have an energy less than the rest energy of an electron, there are still plenty of gamma-ray photons out there carrying more than 0.5 MeV of energy. In the limit $h\nu \gg m_e c^2$, it is more useful to think of light as being a stream of photons, rather than an electromagnetic wave. The scattering of a high-energy photon from an electron is known as *Compton scattering.* As Compton pointed out, a photon has both an energy $\varepsilon = h\nu$ and a momentum $p = \varepsilon/c = h\nu/c$. A collision between a photon and an electron must conserve both energy and momentum. If we treat it as a classical problem, it's a simple "billiard ball" collision.

For simplicity of calculation, let's place ourselves in a frame of reference in which the electron is initially at rest, and orient our cartesian axes so that the photon is initially moving along the x axis (Figure 3). The initial momentum of the two-particle system is

$$\vec{p}_i = \vec{p}_{\text{elec},i} + \vec{p}_{\text{phot},i} = 0 + (h\nu_i/c)\hat{e}_x \ . \tag{14}$$

The initial energy of the system is

$$\varepsilon_i = \varepsilon_{\text{elec},i} + \varepsilon_{\text{phot},i} = m_e c^2 + h\nu_i \ . \tag{15}$$

After scattering, the photon leaves at an angle θ (see Figure 3) with a new, lower frequency ν_f. The electron flies off at an angle Φ with a velocity u_f (and hence a dimensionless velocity $\beta_f = u_f/c$ and Lorentz factor $\gamma_f = (1-\beta_f^2)^{1/2}$). The final momentum of the system is

$$\begin{aligned}
\vec{p}_f &= \vec{p}_{\text{elec},f} + \vec{p}_{\text{phot},f} && (16)\\
&= \gamma_f m_e v \cos\Phi \hat{e}_x + \gamma_f m_e v \sin\Phi \hat{e}_y && (17)\\
&\quad +(h\nu_f/c)\cos\theta \hat{e}_x + (h\nu_f/c)\sin\theta \hat{e}_x \ . && (18)
\end{aligned}$$

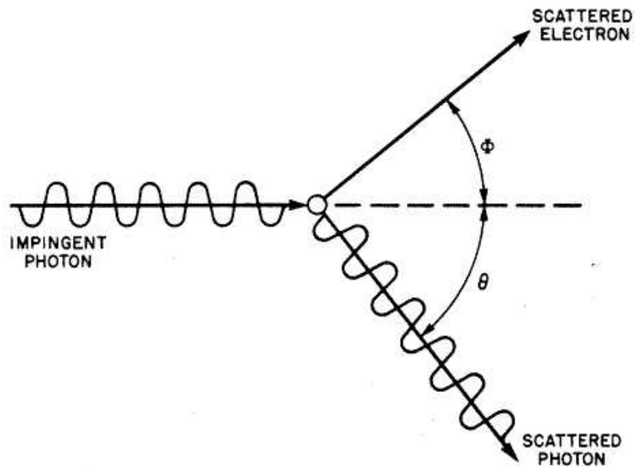

Figure 3: Compton scattering: frame of reference in which electron is initially at rest.

The final energy of the system is

$$\varepsilon_f = \varepsilon_{\mathrm{elec},f} + \varepsilon_{\mathrm{phot},f} = \gamma_f m_e c^2 + h\nu_f \ . \tag{19}$$

With a bit of algebra, we can find ν_f in terms of ν_i and θ. First of all, the natural unit of energy in this problem is the rest energy of the electron. In these units, we can write the initial and final energy of the photon as

$$X_i \equiv \frac{h\nu_i}{m_e c^2} \tag{20}$$

$$X_f \equiv \frac{h\nu_f}{m_e c^2} \ . \tag{21}$$

Conservation of momentum in the x direction tells us

$$\gamma_f \beta_f \cos\Phi = X_i - X_f \cos\theta \ , \tag{22}$$

Conservation of momentum in the y direction tells us

$$\gamma_f \beta_f \sin\Phi = -X_f \sin\theta \ , \tag{23}$$

and conservation of energy tells us

$$\gamma_f - 1 = X_i - X_f \ . \tag{24}$$

That which the electron gains, the photon loses. It's a zero sum game.

Squaring each of the momentum equations (22 and 23), then adding them together eliminates the angle Φ:

$$\gamma_f^2 \beta_f^2 = X_f^2 + X_i^2 - 2 X_i X_f \cos\theta \ . \tag{25}$$

Since $\beta_f^2 = 1 - 1/\gamma_f^2$, we can rewrite this as

$$\gamma_f^2 - 1 = X_f^2 + X_i^2 - 2 X_i X_f \cos\theta \ . \tag{26}$$

By rearranging, we can write the electron's final Lorentz factor in the form

$$\gamma_f^2 = 1 + (X_i - X_f)^2 + 2 X_i X_f (1 - \cos\theta) \ . \tag{27}$$

However, the energy conservation equation (24) tells us that

$$\gamma_f^2 = [1 + (X_i - X_f)]^2 = 1 + 2(X_i - X_f) + (X_i - X_f)^2 \ . \tag{28}$$

Comparison of equations (27) and (28) reveals that

$$2 X_i X_f (1 - \cos\theta) = 2(X_i - X_f) \ , \tag{29}$$

or

$$X_f = \frac{X_i}{1 + X_i(1 - \cos\theta)} \ . \tag{30}$$

Since $1 - \cos\theta$ is always non-negative, $X_f \leq X_i$. The photon always loses energy, except in the special case $\cos\theta = 1$, when the photon is not deflected at all.

In the low-energy limit of Thomson scattering, the initial energy (in units of $m_e c^2$) is $X_i \ll 1$, and the final energy is $X_f \approx X_i[1 - X_i(1 - \cos\theta)]$. Thus, the energy lost by the photon and gained by the electron, is only

$$\Delta X = X_i - X_f \approx X_i^2 (1 - \cos\theta) \ll X_i \ . \tag{31}$$

Thus, the transfer of energy from the photon to the electron is small. The frequency of the light can be approximated as being constant, and the velocity imparted to the electron is highly non-relativistic.

In the extremely high energy limit, where $X_i(1 - \cos\theta) \gg 1$, the final energy is $X_f \approx (1 - \cos\theta)^{-1}$, independent of the initial photon energy, and depending only on angle. The energy lost by the photon (and gained by the electron) is

$$\Delta X = X_i - X_f \approx X_i \gg 1 \ . \tag{32}$$

Thus, the final state of the electron (in the frame of reference where it was initially at rest) is highly relativistic. Thus, one way of giving an electron $\gamma \gg 1$ is to hit it with a photon that has $h\nu_i \gg m_e c^2$.

Earlier, I defined X_i and X_f as dimensionless photon energies; for instance,

$$X_i \equiv \frac{h\nu_i}{m_e c^2} \ . \tag{33}$$

However, since $\nu_i = c/\lambda_i$, we can also write

$$X_i \equiv \frac{h}{m_e c}\frac{1}{\lambda_i} = \frac{\lambda_C}{\lambda_i} \ , \tag{34}$$

where $\lambda_C = h/(m_e c) \approx 2.43 \times 10^{-10}\,\mathrm{cm}$ is the *Compton wavelength* of the electron. In terms of wavelengths, the energy formula for Compton scattering (equation 30) can be written in the form

$$\lambda_f - \lambda_i = \lambda_C (1 - \cos\theta) \ . \tag{35}$$

That is, the shift in wavelength produced by Compton scattering will be comparable in magnitude to the Compton wavelength.[2]

So far, I've been able to compute the properties of Compton scattering without resorting to quantum mechanics. However, a correct calculation of the cross-section for Compton scattering requires the use of quantum mechanics. Unfortunately, it's an ugly calculation (even Rybicki and Lightman refuse to show it). However, in the 1920s, a brave pair of physicists, Oskar Klein and Yoshio Nishina, tackled the problem. The cross-section that they calculated is named the *Klein-Nishina cross-section* in their honor. The full formula is given in Rybicki and Lightman. I'll just look at the limiting cases of low photon energy and high photon energy. In the limit $X_i \ll 1$, the electron's cross-section is

$$\sigma_{KN} \approx \sigma_T (1 - 2X_i) \ , \tag{36}$$

[2]Note also that $\Delta\lambda \equiv \lambda_f - \lambda_i$ is independent of λ_i; it depends only on the scattering angle θ.

which reduces to the Thomson cross-section σ_T as $X_i \to 0$. In the limit $X_i \gg 1$, the electron's cross-section is

$$\sigma_{KN} \approx \sigma_T \frac{3}{8x} \left[\ln(2x) + 1/2 \right] \ . \tag{37}$$

Klein and Nishina also calculated that the differential cross-section is strongly peaked in the forward direction ($\theta \sim 0$) in the limit of high photon energy.