Astronomy 825: Radiative Gas Dynamics Winter 2011

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Contents

Notes on the Notes

These lecture notes are the result of my having taught Astronomy 825 several times, starting in 1993, when my knowledge of radiative gas dynamics was nearly non-existent. The most valuable resource I found while cramming to teach the course was the two-volume *Physics of Astrophysics* set by Frank Shu. Volume I of Shu's work is titled *Radiation* and Volume II is titled *Gas Dynamics*, and between them they cover more than you might want to know about radiative gas dynamics. More recent works that I have consulted while updating these notes are *An Introduction to Astrophysical Fluid Dynamics* by Michael Thompson and *Principles of Astrophysical Fluid Dynamics* by C. J. Clarke and R. F. Carswell.

In a way, these (not-a-textbook) notes are a companion to Rick Pogge's (not-a-textbook) notes for Astronomy 871, *Physics of the Interstellar Medium*. I have attempted to make notation and units consistent between the two sets of notes. Thus, I adopt cgs units, supplemented by angstroms (Å), electron volts (eV), and parsecs (pc).

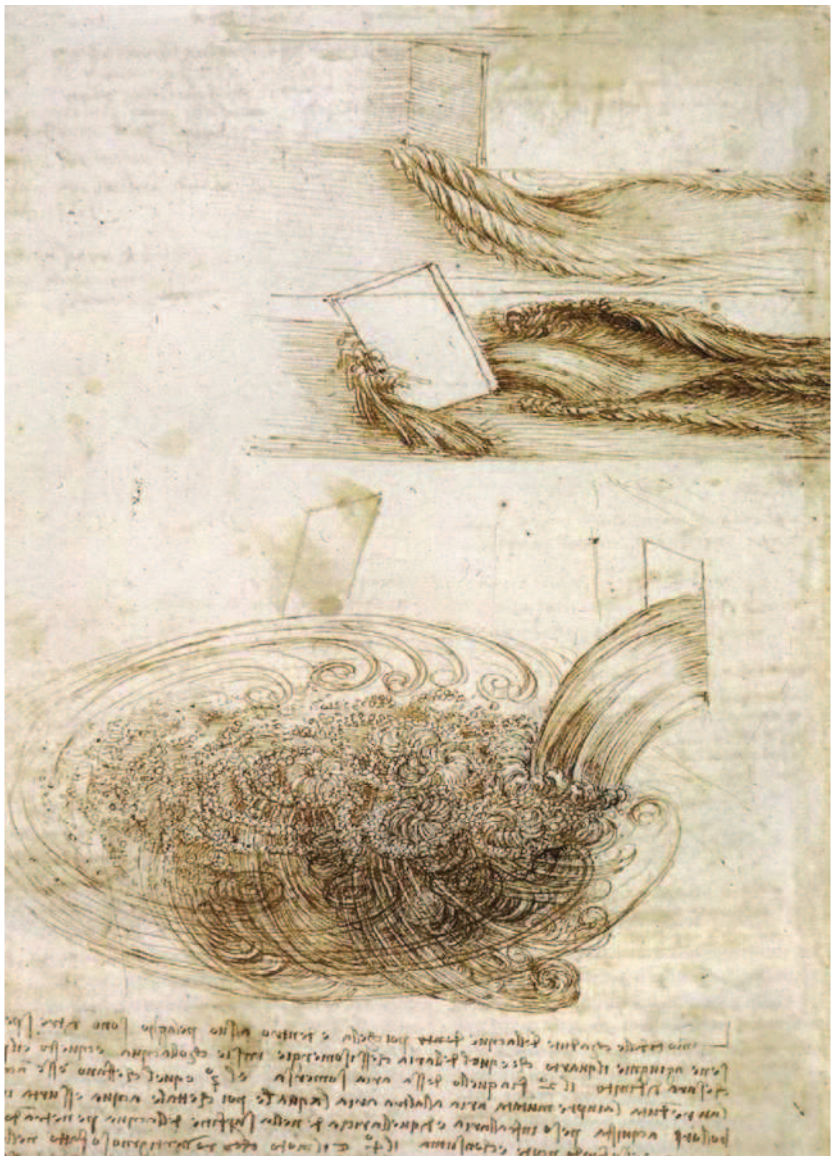

Figure 1: Turbulent fluids as drawn by Leonardo da Vinci.

Chapter 1

Fundamentals of Gas Dynamics

Gas dynamics is the study of continuous compressible fluids in motion. Real gases are not perfectly continuous, since they are made of individual particles: atoms, molecules, ions, and/or electrons. However, a gas can be adequately approximated as a continuous fluid when

$$\lambda \ll L \tag{1.1}$$

where λ is the mean free path of the gas particles, and L is the characteristic size of the system. The mean free path in a gas of neutral particles is

$$\lambda = \frac{1}{n\sigma} \tag{1.2}$$

where n is the number density of particles, and σ is the cross section for collisions. A typical cross section for atoms or small molecules is $\sigma \sim 10^{-15}\,\mathrm{cm}^2$, or about 1 gigabarn. In air at room temperature, $n \sim 10^{19}\,\mathrm{cm}^{-3}$, and hence $\lambda \sim 10^{-4}\,\mathrm{cm} \sim 1\,\mu\mathrm{m}$. Thus, a volume of air that is larger than several microns on a side can be treated as a continuous fluid. In the interstellar medium (ISM), particle number densities are much lower than in the Earth's atmosphere. In a molecular cloud, $n \sim 1000\,\mathrm{cm}^{-3}$, and hence $\lambda \sim 10^{12}\,\mathrm{cm} \sim 0.07\,\mathrm{AU}$. In the warm neutral medium, $n \sim 0.5\,\mathrm{cm}^{-3}$, and hence $\lambda \sim 2 \times 10^{15}\,\mathrm{cm} \sim 100\,\mathrm{AU} \sim 6 \times 10^{-4}\,\mathrm{pc}$.

The situation is more complicated in a plasma. Consider a gas of fully ionized hydrogen. The effective radius of interaction r_e for a free electron can be found by setting the magnitude of its potential energy at a distance r_e from an electron or proton equal to its thermal kinetic energy: is found by setting

$$\frac{e^2}{r_e} \sim m_e v_e^2 \ . \tag{1.3}$$

Since $m_e v_e^2 \sim kT$, where T is the kinetic temperature of the free electrons, we can write

$$r_e \sim \frac{e^2}{m_e v_e^2} \sim \frac{e^2}{kT} \ . \tag{1.4}$$

The cross section is thus

$$\sigma \sim \pi r_e^2 \sim \frac{\pi e^4}{k^2 T^2} \ , \tag{1.5}$$

and the mean free path for an electron is

$$\lambda \sim \frac{k^2 T^2}{\pi e^4 n} \ . \tag{1.6}$$

(Note: a more accurate calculation would contain the Coulomb logarithm $\ln \Lambda$, but this is good enough for an order-of-magnitude estimate.) In the hot ionized interstellar medium, $T \sim 10^6\,\mathrm{K}$ and $n \sim 3 \times 10^{-3}\,\mathrm{cm}^{-3}$. The mean free path is thus $\lambda \sim 4 \times 10^{19}\,\mathrm{cm} \sim 10\,\mathrm{pc}$.

It's useful at this point to provide a sketch of the different components of the ISM. **Molecular Clouds** consist mainly of molecular gas (H_2, CO, etc.) along with stubbornly solitary He atoms. Typical temperatures and number densities are

$$T \sim 15\,\mathrm{K} \text{ and } n \gtrsim 1000\,\mathrm{cm}^{-3} \ . \tag{1.7}$$

The **Cold Neutral Medium** consists mainly of neutral atomic gas (H I, He I, etc.) Typical temperatures and number densities are

$$T \sim 90\,\mathrm{K} \text{ and } n \sim 50\,\mathrm{cm}^{-3} \ . \tag{1.8}$$

The **Warm Neutral Medium** consists mainly of neutral atomic gas, but at a higher temperature and lower density than the Cold Neutral Medium. Typical temperatures and number densities are

$$T \sim 8000\,\mathrm{K} \text{ and } n \sim 0.5\,\mathrm{cm}^{-3} \ . \tag{1.9}$$

The **Warm Ionized Medium** consists of partially ionized gas (H I, He I, H II, He II, He III, etc.) Typical temperatures and number densities are

$$T \sim 10^4\,\mathrm{K} \text{ and } n \sim 0.1\,\mathrm{cm}^{-3} \ . \tag{1.10}$$

The **Hot Ionized Medium** consists of ionized gas at very high temperatures. Typical temperatures and number densities are

$$T \gtrsim 10^6\,\mathrm{K} \text{ and } n \lesssim 0.003\,\mathrm{cm}^{-3} \,. \tag{1.11}$$

Although the different components of the ISM are not in perfect pressure equilibrium, it is notable that the pressures implied by the values of n and T given above are all within an order of magnitude of each other, with a typical value

$$P = nkT \sim 3 \times 10^{-13}\,\mathrm{dyne\ cm}^{-2} \sim 3 \times 10^{-19}\,\mathrm{atm} \,. \tag{1.12}$$

The "atmosphere" (atm) is approximately equal to the pressure of the Earth's atmosphere at sea level. The ISM is obviously very low in pressure compared to the air around us; thus, our intuitions about gas dynamics, which are largely based on the behavior of air, may not always apply to the ISM.

The intergalactic medium (IGM) is typically at densities much lower than even the Hot Ionized Medium of the ISM. The **Warm/Hot Intergalactic Medium (WHIM)** has typical temperatures and number densities

$$T \sim 10^6\,\mathrm{K} \text{ and } n \sim 5 \times 10^{-6}\,\mathrm{cm}^{-3} \,. \tag{1.13}$$

The **intracluster medium (ICM)** in dense clusters of galaxies can have

$$T \sim 10^8\,\mathrm{K} \text{ and } n \sim 10^{-3}\,\mathrm{cm}^{-3} \,. \tag{1.14}$$

Although the intracluster medium is thousands of times higher in pressure than the WHIM, it is kept in hydrostatic equilibrium by the gravitational pull of the dark matter in clusters.

Much of the ISM is flowing out of, or accreting onto, compact objects. Examples of **outflow** are blast waves, winds, and jets. Examples of **accretion** are cooling flows and accretion disks. Interesting things also happen where one phase of the interstellar medium meets another. A **shock front** is a boundary between low density, supersonic flow and high density, subsonic flow. An **ionization front** is a boundary between ionized matter and neutral matter.

In many astrophysical contexts, radiative effects influence the flow of gas. Molecular clouds can absorb and emit radiation by rotational and vibrational transitions. Sufficiently energetic radiation can photodissociate the molecules; the dissociation energy of H_2 is 4.48 eV, corresponding to a wavelength of 2770Å. Neutral atomic gas can absorb and emit radiation through bound-bound electronic transitions. Sufficiently energetic radiation can photoionize the atoms; the ionization energy of H is 13.6 eV, corresponding to a wavelength of 912Å. Ionized gas can emit radiation through bremsstrahlung (free-free transitions) and radiative recombination. The electrons and ions can also interact with photons via Compton scattering. In the presence of a magnetic field, electrons can lose energy by synchrotron radiation.

Radiative transfer in the interstellar medium isn't always pretty. To make life simpler, let's start by considering the dynamics of gas that is *not* radiating. A good place to begin is with the **Boltzmann Equation**.[1] At a given time t, any particle of the gas has a position $\vec{x}$ and a velocity $\vec{v}$. So, if we live in a Newtonian deterministic universe, we can compute the development of a system if we know all the positions and velocities at a given time and if we know the external forces (gravitational, electrostatic, magnetic, etc.) working on the particles. This fully deterministic approach, however, becomes a major pain in the neck as the number of gas particles becomes large. If you are dealing with many, many gas particles, a probabilistic approach becomes much more practical and much more useful.

Let $f(\vec{x}, \vec{v}, t)d^3xd^3v$ be the probability of finding a gas particle within the six-dimensional phase space volume d^3xd^3v at position $\vec{x}$ with a velocity $\vec{v}$ at time t. The six-dimensional space mapped out by $\vec{x}$ and $\vec{v}$ is known as **phase space**. The three-dimensional space mapped by $\vec{x}$ is known as **position space**; the three-dimensional space mapped by $\vec{v}$ is known as **velocity space**. The function $f(\vec{x}, \vec{v}, t)$ is known as the **distribution function**.

The distribution function is normalized so that

$$\int\int f(\vec{x}, \vec{v}, t)d^3xd^3v = N \ , \tag{1.15}$$

where N is the total number of particles in the system. If we choose, we can also define a *mass* distribution function

$$f_m(\vec{x}, \vec{v}, t) = mf(\vec{x}, \vec{v}, t) \ , \tag{1.16}$$

where m is the mass of an individual particle. If n types of particle are present, each with a different mass, the *total* mass distribution function is

$$f_m = \sum_{k=1}^{n} m_k f_k(\vec{x}, \vec{v}, t) \ , \tag{1.17}$$

[1]The Boltzmann Equation also plays a key role in the dynamics of stellar systems.

where m_k is the mass of the $k^{\rm th}$ type of particle. If particles are neither created nor destroyed,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\dot{x}_i \frac{\partial f}{\partial x_i} + \dot{v}_i \frac{\partial f}{\partial v_i} \right) = \frac{df}{dt}\Big|_c \ . \tag{1.18}$$

The factor on the right hand side of the above equation is the rate at which particles are bumped into a phase space volume by collisions with other particles. The word "collisions", when used in this context, can be misleading, since it implies that the individual particles behave like billiard balls, interacting only when they touch each other. In an ionized gas, the individual charged particles interact through long-range electromagnetic forces. A "collision", in that case, is a close encounter between charged particles during which the electromagnetic force changes their direction of motion by $\sim 90^\circ$ or more. Similarly, in stellar dynamics, a "collision" between stars is a close encounter during which gravity changes their direction of motion by $\sim 90^\circ$ or more.

We may rewrite equation (1.18) by making use of the fact that $\dot{\vec{x}} = \vec{v}$ and that $\dot{\vec{v}} = \vec{g}$, where $\vec{g}$ is the acceleration of a particle at position $\vec{x}$ with velocity $\vec{v}$. The gravitational force acting on the particle results in an acceleration

$$\vec{g} = -\vec{\nabla}\Phi \ , \tag{1.19}$$

where the potential $\Phi(\vec{x})$ has contributions both from the self-gravity of the system of particles and from any external gravitational fields that may be present.[2] The electromagnetic force acting on the particle results in an acceleration

$$\vec{g} = \frac{q}{m} \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \ , \tag{1.20}$$

where q is the charge of the particle, m is its mass, $\vec{E}$ is the electric field at the particle's location, and $\vec{B}$ is the magnetic field there.

The phase space continuity equation can now be written in the form

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(v_i \frac{\partial f}{\partial x_i} + g_i \frac{\partial f}{\partial v_i} \right) = \frac{df}{dt}\Big|_c \ . \tag{1.21}$$

[2]These notes are Newtonian. Unless I make an explicit statement to the contrary, all motions are assumed to be non-relativistic, and gravity is assumed to be an inverse-square-law force.

This equation is known as the **Boltzmann equation**.

In equilibrium, particles are bumped into an element of phase space at the same rate that they are bumped out, and the collisional term on the right-hand side is equal to zero. If the collisional term is ignored, the resulting equation is called the **collisionless Boltzmann equation**. The collisionless Boltzmann equation is the basis of stellar dynamics and of hydrodynamics. However, we are often unable to determine the full velocity distribution at every point in position space. What we usually want to deal with is the number density or mass density in position space, plus the mean (or bulk) velocity of the gas, plus its velocity dispersion.

Now, let's take the moments of the collisionless Boltzmann equation. Let's start with f_m (the mass density in phase space) rather than f (the number density in phase space), because it is useful to think about the conservation of mass (m), the conservation of momentum ($m \times \vec{v}$), and the conservation of kinetic energy ($m \times v^2/2$). The mass density in position space is found by integrating over all velocities:

$$\rho(\vec{x}, t) = \int f_m(\vec{x}, \vec{v}, t) d^3v \ . \tag{1.22}$$

For any measurable quantity Q, the mass-weighted average value at a position $\vec{x}$ at time t is

$$\langle Q \rangle = \frac{1}{\rho} \int Q f_m d^3v \ . \tag{1.23}$$

Integrate the collisionless Boltzmann equation over all velocities:

$$\frac{\partial}{\partial t} \int f_m d^3v + \sum_i \frac{\partial}{\partial x_i} \int v_i f_m d^3v = -\sum_i g_i \int \frac{\partial f_m}{\partial v_i} d^3v \ . \tag{1.24}$$

The partial differentials have been taken outside the integrals since $\vec{x}$, $\vec{v}$, and t are all independent variables. The right hand side of the above equation vanishes by the divergence theorem, as long as $f_m \to 0$ as $v \to \infty$.

Then

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial}{\partial x_i} (\rho \langle v_i \rangle) = 0 \ . \tag{1.25}$$

Now introduce a new symbol:

$$\vec{u} \equiv \langle \vec{v} \rangle \ . \tag{1.26}$$

The velocity $\vec{u}$ is the **bulk velocity** at a given point in space. The velocity $\vec{v}$ of a particular particle may then be broken into two components:

$$\vec{v} = \vec{u} + \vec{w} \ , \tag{1.27}$$

where $\vec{w}$ is the **random velocity**.

In vector form,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \ . \tag{1.28}$$

This is the **continuity equation**. It states that mass is conserved. Moreover, the flow of matter is continuous; mass does not disappear at one point and simultaneously appear at another point some distance away.

The continuity equation is a single equation in four unknowns (ρ and the three components of $\vec{v}$). To obtain more equations, let's find the next higher moment of the collisionless Boltzmann equation; multiply by v_j and integrate over all velocities.

$$\frac{\partial}{\partial t} \int v_j f_m d^3 v + \sum_i \frac{\partial}{\partial x_i} \int v_j v_i f_m d^3 v = - \sum_i g_i \int v_j \frac{\partial f}{\partial v_i} d^3 v \ . \tag{1.29}$$

Rewriting the integrand on the right hand side, we have

$$\frac{\partial}{\partial t}(\rho \langle v_j \rangle) + \sum_i \frac{\partial}{\partial x_i}(\rho \langle v_j v_i \rangle) = - \sum_i g_i \int \left[\frac{\partial}{\partial v_i}(v_j f) - \delta_{ij} f \right] d^3 v \ . \tag{1.30}$$

Now we use the decomposition $\vec{v} = \vec{u} + \vec{w}$ on the left, and the divergence theorem on the right (assuming that $f_m v \to 0$ as $v \to \infty$). This tells us that

$$\frac{\partial}{\partial t}(\rho u_j) + \sum_i \frac{\partial}{\partial x_i}(\rho u_i u_j + \rho \langle w_i w_j \rangle) = \rho g_j \ . \tag{1.31}$$

This is the equation of momentum conservation. It is convenient to write the pressure tensor $\rho \langle w_i w_j \rangle$ in the form

$$\rho \langle w_i w_j \rangle = P \delta_{ij} - \pi_{ij} \ , \tag{1.32}$$

where P is the **pressure**,

$$P \equiv \frac{1}{3} \rho \langle |\vec{w}|^2 \rangle \ , \tag{1.33}$$

and π_{ij} is the **viscous stress tensor**

$$\pi_{ij} \equiv P \delta_{ij} - \rho \langle w_i w_j \rangle \ . \tag{1.34}$$

It is useful to make this division because in some cases the viscous stress tensor can be ignored. In other cases, the viscous stress tensor can be computed in terms of the shear of the bulk velocity.

The equation for momentum conservation using the formalism that we have adopted:

$$\frac{\partial}{\partial t}(\rho u_j) + \sum_i \frac{\partial}{\partial x_i}(\rho u_i u_j + P\delta_{ij} - \pi_{ij}) = \rho g_j \ . \tag{1.35}$$

The equation can be written in tensor form:

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u}\vec{u} + P \overleftrightarrow{I} - \overleftrightarrow{\pi}) = \rho \vec{g} \ . \tag{1.36}$$

In the above equation, $\overleftrightarrow{I}$ is the unit matrix:

$$\overleftrightarrow{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{1.37}$$

The tensor product of the velocities is

$$\vec{u}\vec{u} = \begin{pmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{pmatrix} \ . \tag{1.38}$$

Finally, the viscous stress tensor is the symmetric traceless tensor

$$\overleftrightarrow{\pi} = \begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{12} & \pi_{22} & \pi_{23} \\ \pi_{13} & \pi_{23} & -\pi_{11} - \pi_{22} \end{pmatrix} \ . \tag{1.39}$$

The tensor form of the momentum conservation equation tells us that the time derivative of a conserved quantity plus the divergence of a flux is equal to a source term; this is the standard form of a conservation equation.

Let's go one step further and look at the conservation of kinetic energy. Multiply the collisionless Boltzmann equation by v^2, and integrate over all velocities.

$$\frac{\partial}{\partial t}\int v^2 f_m d^3v + \sum_i \frac{\partial}{\partial x_i}\int v_i v^2 f_m d^3v = -\sum_i g_i \int v^2 \frac{\partial f_m}{\partial v_i} d^3v \ . \tag{1.40}$$

Rewriting the integrand on the right hand side,

$$\frac{\partial}{\partial t}(\rho\langle v^2\rangle) + \sum_i(\rho\langle v_i v^2\rangle) = -\sum_i g_i \int \left[\frac{\partial}{\partial v_i}(v^2 f_m) - 2v_i f_m\right] d^3v \ . \tag{1.41}$$

Breaking the velocity into its ordered and random components ($\vec{v} = \vec{u} + \vec{w}$) we find

$$\begin{aligned}&\frac{\partial}{\partial t}(\rho u^2 + \rho\langle w^2\rangle) + \qquad (1.42)\\ &\sum_i \frac{\partial}{\partial x_i}(\rho[u_i u^2 + u_i\langle w^2\rangle + 2\sum_j u_j\langle w_i w_j\rangle + \langle w_i w^2\rangle]) = 2\sum_i g_i \rho u_i \ .\end{aligned}$$

Some more definitions: the **specific internal energy** of a monatomic gas is

$$\varepsilon \equiv \frac{1}{2}\langle w^2\rangle \ ; \tag{1.43}$$

this is just the energy per unit mass contributed by the random internal motions. Note that for a monatomic gas

$$\varepsilon = \frac{3}{2}\frac{P}{\rho} \ . \tag{1.44}$$

Next, the **conductive heat flux** is

$$\vec{F} \equiv \frac{1}{2}\rho\langle \vec{w} w^2\rangle \ . \tag{1.45}$$

If the distribution of random velocities $\vec{w}$ is symmetric about zero, then the conductive heat flux $\vec{F}$ will vanish. However, if $\vec{w}$ has a skewed distribution, then the "hot" particles (those with large random velocities) will have a net drift relative to the "cold" particles.

Using the newly defined quantities, the equation of energy conservation is

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho u^2 + \rho\varepsilon) + \vec{\nabla}\cdot\left[(\frac{1}{2}\rho u^2 + P + \rho\varepsilon)\vec{u} - \overleftrightarrow{\pi}\cdot\vec{u} + \vec{F}\right] = \rho\vec{u}\cdot\vec{g} \ . \tag{1.46}$$

By combining the mass conservation equation with the momentum conservation equation, we can write down an equation for the conservation of the bulk kinetic energy ($\rho u^2/2$):

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho u^2) + \vec{\nabla}\cdot(\frac{1}{2}\rho u^2\vec{u}) = \rho\vec{u}\cdot\vec{g} - \vec{u}\cdot\vec{\nabla}P + \vec{u}\cdot(\vec{\nabla}\cdot\overleftrightarrow{\pi}) \ . \tag{1.47}$$

Figure 1.1: Jim and Huckleberry Finn, drifting along, are Lagrangian observers; someone on the bank would be an Eulerian observer of the river.

By subtracting this equation from the equation for the conservation of the total energy, we find the **internal energy equation**:

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \vec{\nabla} \cdot (\rho\varepsilon\vec{u}) = -P\vec{\nabla} \cdot \vec{u} - \vec{\nabla} \cdot \vec{F} + \Psi \ , \tag{1.48}$$

where

$$\Psi \equiv \sum_{i,j} \pi_{ij} \frac{\partial u_i}{\partial x_j} \ . \tag{1.49}$$

The function Ψ is the **rate of viscous dissipation**; viscosity converts bulk kinetic energy into internal energy.

The partial time derivative of a function Q, which we have written as $\partial Q/\partial t$, is the rate of change as viewed by an observer at a fixed coordinate position, who is watching the gas flow by. Such an observer is called an **Eulerian** observer. An alternative point of view is that of an observer who is moving along with the bulk flow of the gas. Such an observer is called a **Lagrangian** observer.

The Lagrangian time derivative,

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \vec{u} \cdot \vec{\nabla} Q \ , \tag{1.50}$$

is the rate of change of Q as seen by a Lagrangian observer, who is moving along with a designated parcel of gas. Note that the Lagrangian time derivative of a vector is $(\vec{u} \cdot \vec{\nabla})\vec{Q} = (\vec{u} \cdot \vec{\nabla} Q_x)\hat{\imath} + (\vec{u} \cdot \vec{\nabla} Q_y)\hat{\jmath} + (\vec{u} \cdot \vec{\nabla} Q_z)\hat{k})$, when written out in Cartesian coordinates. For a spherically symmetric system, where all quantities are functions only of t and r (the radial coordinate), it is useful to write the Lagrangian time derivatives in spherical coordinates:

$$\frac{DQ(r,t)}{Dt} = \frac{\partial Q}{\partial t} + u_r \frac{\partial Q}{\partial r} \ . \tag{1.51}$$

and, for vectors,

$$\frac{D\vec{Q}(r,t)}{Dt} = \frac{\partial \vec{Q}}{\partial t} + u_r \frac{\partial \vec{Q}}{\partial r} \ . \tag{1.52}$$

In a Lagrangian form, the **continuity equation** is

$$\frac{D\rho}{Dt} = -\rho \vec{\nabla} \cdot \vec{u} \ . \tag{1.53}$$

The **momentum equation** is

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla} P + \frac{1}{\rho}\vec{\nabla} \cdot \overleftrightarrow{\pi} + \vec{g} \ . \tag{1.54}$$

The acceleration is provided by the pressure gradient, the viscous drag, and by the gravitational and electromagnetic forces that we have included in $\vec{g}$.

The **internal energy equation** is

$$\frac{D\varepsilon}{Dt} = -\frac{P}{\rho}\vec{\nabla} \cdot \vec{u} - \frac{1}{\rho}\vec{\nabla} \cdot \vec{F} + \frac{1}{\rho}\Psi \ . \tag{1.55}$$

The change in internal energy is given by a PdV work term, a heat conduction term, and a viscous heating term.

Equations (1.53), (1.54), and (1.55) are valid for *nonradiative* gas dynamics. Since this is a course on radiative gas dynamics, we must take into account the effect of radiation on the momentum and energy of the gas. To the right hand side of the internal energy equation, we must add the term $(\Gamma - \Lambda)/\rho$, where Γ is the volumetric heating rate (or "gain"), and Λ is the volumetric cooling rate (or "loss").[3]

[3] In certain cases, such as systems near the Eddington Limit, radiative pressure has an important effect on the motion of the gas, and must be included in the momentum conservation equation.

The equations for conservation of mass, momentum, and energy provide *five* equations in all. How many unknowns are there? There's the density ρ, the pressure P, the bulk velocity $\vec{u}$ (3 unknowns), the conductive heat flux $\vec{F}$ (3 unknowns), and the viscous stress tensor $\overleftrightarrow{\pi}$ (5 independent components). The internal energy, for a monatomic gas, is given by the relation $\varepsilon = 3P/(2\rho)$, the rate of viscous dissipation can be computed from $\overleftrightarrow{\pi}$ and $\vec{u}$, the acceleration due to self-gravity is given by Poisson's equation

$$\vec{\nabla} \cdot \vec{g} = -4\pi G\rho \ , \tag{1.56}$$

and the acceleration from any external sources is assumed to be known.

So, we have 5 equations and 13 unknowns. The clever way to break out of the hierarchy of equations is to express the viscous stress tensor $\overleftrightarrow{\pi}$ and the conductive heat flux $\vec{F}$ in terms of ρ, P, and $\vec{u}$. This will leave us with 5 equations and 5 unknowns. Once we specify our initial conditions and boundary conditions, we can proceed to solve the equations.

Chapter 2

Viscosity, Heat Conduction, and Other Complications

The task at hand is to find expressions for the viscous stress tensor $\overleftrightarrow{\pi}$ and the conductive heat flux $\vec{F}$ in terms of the density, pressure, and bulk velocity of a gas. We will start with an ideal gas.

The distribution of random velocities for an ideal gas (otherwise known as a "perfect gas") is the Maxwellian distribution

$$f(\vec{w})d^3w = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mw^2}{2kT}\right) d^3w \ , \tag{2.1}$$

where k is the Boltzmann constant, m is the particle mass, and T is the kinetic temperature. The mean square random velocity is then $\langle w^2 \rangle = 3kT/m$. Using equation (1.33), we find that the pressure is given by the familiar ideal gas law

$$P = \frac{\rho}{m}kT \ . \tag{2.2}$$

For the Maxwellian distribution, $\langle w_i w_j \rangle = 0$ when $i \neq j$; hence, all elements of $\overleftrightarrow{\pi}$ vanish. Furthermore, $\langle w^2 w_i \rangle = 0$, so the conductive heat flux also vanishes. Ideal gases are delightfully simple: no viscosity, no heat conduction.

The conservation laws for an ideal gas take the simplified form

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla} \cdot \vec{u} \tag{2.3}$$

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla}P + \vec{g} \tag{2.4}$$

$$\frac{D\varepsilon}{Dt} = -\frac{P}{\rho}\vec{\nabla}\cdot\vec{u} \ . \tag{2.5}$$

These three equations are known collectively as the **Euler equations**. In conjunction with the ideal gas law

$$\varepsilon = \frac{3P}{2\rho} = \frac{3kT}{2m} \ , \tag{2.6}$$

they describe the motions of an inviscid, ideal gas of point masses.[1] Combining the continuity and energy conservation equations, we see that for an ideal gas,

$$\frac{D\varepsilon}{Dt} = \frac{P}{\rho^2}\frac{D\rho}{Dt} = -P\frac{DV}{Dt} \ , \tag{2.7}$$

where $V \equiv 1/\rho$ is the specific volume of the gas. The change in internal energy of a ideal gas is equal to the PdV work that is done on the gas.

The first law of thermodynamics states that

$$Tds = d\varepsilon + PdV \ , \tag{2.8}$$

where T and P are the temperature and pressure of the gas, s is the specific entropy, ε is the specific internal energy, and V is the specific volume. Since, for an ideal gas,

$$\frac{D\varepsilon}{Dt} = -P\frac{DV}{Dt} \ , \tag{2.9}$$

it is necessary that

$$T\frac{Ds}{Dt} = 0 \ . \tag{2.10}$$

An ideal gas, in the absence of heat sources and sinks, undergoes only **adiabatic** processes. (That is, its entropy remains constant.)

If a gas is compelled to have a constant volume (by being enclosed in a rigid box, for instance), the first law of thermodynamics reduces to $dq = d\varepsilon$, where dq is the heat added. The **specific heat** at constant volume is then

$$c_V \equiv \left(\frac{\partial q}{\partial T}\right)_V = \left(\frac{\partial \varepsilon}{\partial T}\right)_V \ . \tag{2.11}$$

However, for an ideal gas, $\varepsilon(V,T) = \varepsilon(T)$. Thus, we may write, quite generally,

$$d\varepsilon = c_V dT \tag{2.12}$$

[1] Inviscid = having no viscosity.

and

$$dq = c_V dT + PdV \ . \tag{2.13}$$

Now consider gas that is kept at a constant pressure. The ideal gas law tells us that

$$PdV = \frac{k}{m} dT \tag{2.14}$$

when pressure is kept constant, and hence that

$$c_P \equiv \left(\frac{\partial q}{\partial T} \right)_P = c_V + \frac{k}{m} \ . \tag{2.15}$$

Note that $c_P > c_V$; when pressure is held constant, some of the added heat goes into PdV work instead of into internal energy.

The **adiabatic index** of a gas is defined as $\gamma \equiv c_P/c_V$. For an ideal gas of point particles, $\varepsilon = (3kT)/(2m)$, $c_V = (3k)/(2m)$, $c_P = (5k)/(2m)$ and $\gamma = 5/3$. The adiabatic index for a gas of diatomic molecules is $\gamma = 7/5$. The adiabatic index is a function of the number of degrees of freedom of the particles; diatomic molecules have rotational degrees of freedom that are not present for point masses. The internal energy ε, if the gas particles are not spherical atoms, also includes rotational energy in addition to the translational energy. In general, the internal energy is

$$\varepsilon = \frac{1}{\gamma - 1} \frac{kT}{m} = \frac{1}{\gamma - 1} \frac{P}{\rho} \ . \tag{2.16}$$

Now consider an adiabatic process, for which the first law of thermodynamics may be written

$$\begin{aligned} d\varepsilon + PdV &= 0 && (2.17) \\ c_V dT - \frac{P}{\rho^2} d\rho &= 0 \ . && (2.18) \end{aligned}$$

For an ideal gas,

$$dT = \frac{mP}{k\rho} \left(\frac{dP}{P} - \frac{d\rho}{\rho} \right) \ . \tag{2.19}$$

Combining equations (2.18) and (2.19),

$$c_V \frac{mP}{k\rho} \left[\frac{dP}{P} - \left(\frac{c_V + k/m}{c_V} \right) \frac{d\rho}{\rho} \right] = 0 \ . \tag{2.20}$$

Thus, $dP/P = \gamma(d\rho/\rho)$, and

$$P(\rho) = P_0(\rho/\rho_0)^\gamma \ . \tag{2.21}$$

A gas that has an equation of state in this form ($P \propto \rho^\gamma$) is known as a **polytrope**.

An ideal gas has no viscosity. However, real gases aren't perfectly ideal, and in many astrophysical applications (such as accretion disks) viscosity is important. To handle viscosity in a relatively simple manner, we need to express the tensor $\overleftrightarrow{\pi}$ in terms of the bulk velocity $\vec{u}$. First, viscous frictional forces will occur when two fluid elements move relative to each other. Hence, $\overleftrightarrow{\pi}$ must depend on the spatial derivatives of the velocity, $\partial u_i/\partial x_j$. Second, viscous forces must disappear when the fluid is at rest or is in uniform translational or rotational motion. Third, for small velocity gradients, the elements of $\overleftrightarrow{\pi}$ will be *linearly proportional* to the velocity gradient. (Fluids for which $\pi_{ij} \propto \partial u_i/\partial x_j$ are known as linear fluids, or Newtonian fluids.)

The most general viscous stress tensor that satisfies these three requirements is

$$\pi_{ij} = \mu D_{ij} + \beta \vec{\nabla} \cdot \vec{u} \delta_{ij} \ , \tag{2.22}$$

where μ is the **coefficient of shear viscosity**, β is the **coefficient of bulk viscosity**, and

$$D_{ij} \equiv \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \vec{\nabla} \cdot \vec{u} \delta_{ij} \ . \tag{2.23}$$

The **deformation tensor** D_{ij} vanishes in the case of uniform expansion or contraction. Thus, the shear viscosity term represents pure shear, with no change in volume, while the bulk viscosity term represents pure expansion or contraction.

In the cgs system, the coefficients of viscosity μ and β are measured in 'poises', where 1 poise equals 1 g/cm/sec.[2] At room temperature, the coefficient of shear viscosity for air is $\mu \sim 2 \times 10^{-4}$ poise; for water, $\mu \sim 10^{-2}$ poise.

The coefficient of shear viscosity can be approximately calculated in a fairly simple manner. A gas has a number density n of particles, each with a mass m. The gas has a temperature T, and hence a thermal velocity $v_t = (kT/m)^{1/2}$. In addition to the random thermal velocity of the particles,

[2]The 'poise' is named after the French physiologist Jean Poiseuille.

there is also a bulk velocity u_x in the x direction, where u_x is a function of y. (In other words, there exists a shear, $\partial u_x/\partial y \neq 0$.)

Now consider the plane $y = y_0$. Because of the random velocities of the particles, there is a flux of particles moving *downward* through the plane, with a magnitude per unit area of $\sim nv_t/2$. There is a flux of equal magnitude *upward* through the plane. Thus, there is no net flux of mass density through the plane. However, if $\partial u_x/\partial y > 0$, the downward flux of particles will have a greater momentum in the x direction than will the upward flux of particles. The net result: a transfer of momentum opposite to the direction of the velocity gradient $\partial u_x/\partial y$. The particles that cross the plane from above will have had their last collision at a distance $\sim \lambda$ above the plane. Thus, the downward flux of angular momentum will be

$$P_{yx} \sim \frac{nv_t}{2} m \left[u_x(y_0) + \lambda \frac{\partial u_x}{\partial y} \right] \ . \tag{2.24}$$

Similarly, the upward flux of angular momentum will be

$$P_{yx} \sim \frac{nv_t}{2} m \left[u_x(y_0) - \lambda \frac{\partial u_x}{\partial y} \right] \ . \tag{2.25}$$

The net momentum flux is then

$$P_{yx} \sim -nv_t m \lambda \frac{\partial u_x}{\partial y} \ , \tag{2.26}$$

which tells us that the coefficient of shear viscosity is

$$\mu \sim nv_t m \lambda \sim (mkT)^{1/2}/\sigma \ . \tag{2.27}$$

where σ is the cross sectional area of the gas particles. To lowest order, the coefficient of bulk viscosity, β, is equal to zero. Note that μ is independent of the density of the gas. Note also that the viscosity of a gas increases with temperature, in contrast to the behavior of liquids, in which viscosity generally decreases with temperature.[3] For neutral atomic hydrogen, the coefficient of shear viscosity (including relevant factors of π) is

$$\mu = 6 \times 10^{-3}\,\text{poise} \left(\frac{T}{10^4\,\text{K}} \right)^{1/2} \ . \tag{2.28}$$

[3] The common English simile, "as slow as molasses in January" makes reference to the higher viscosity of molasses at lower temperatures.

Another parameter that is frequently used to describe viscosity is the **kinematic viscosity**,

$$\nu \equiv \mu/\rho \ . \tag{2.29}$$

From equation (2.27), we expect that the kinematic viscosity of a gas should be of order

$$\nu \sim \mu/(nm) \sim v_t \lambda \ , \tag{2.30}$$

where v_t is the thermal speed, and λ is the mean free path of gas particles. The kinematic viscosity of the air around you is $\nu \sim 0.15\,\mathrm{cm^2\,s^{-1}}$. For a gas of neutral atomic hydrogen,

$$\nu \sim 4 \times 10^{21}\,\mathrm{cm^2\,s^{-1}} \left(\frac{T}{10^4 K}\right)^{1/2} \left(\frac{n}{1\,\mathrm{cm^{-3}}}\right) -1 \ . \tag{2.31}$$

A dimensionless number much used by fluid dynamicists is the **Reynolds number**,

$$\mathrm{Re} \equiv \frac{\rho u L}{\mu} = \frac{uL}{\nu} \sim \left(\frac{u}{v_t}\right)\left(\frac{L}{\lambda}\right) \ , \tag{2.32}$$

where L is the typical length scale of the system we're looking at. The Reynolds number is the ratio of the inertial forces ($\sim \rho u^2/L$) to the viscous forces ($\sim \mu u/L^2$). Thus, when $\mathrm{Re} \gg 1$, the viscous forces are negligible. As the scale of interest L becomes smaller and smaller, there is some length scale on which viscosity becomes important. Viscous flow tends to be laminar, while less-viscous flow tends to be turbulent. However, the transition between laminar and turbulent flow is not determined uniquely by the Reynolds number; the geometry of the system is also important. For fluids moving through a straight pipe, to take a well-studied example, flow with $\mathrm{Re} \lesssim 3000$ is laminar and flow with $\mathrm{Re} \gtrsim 3000$ is turbulent. For a dimpled golf ball, the critical Reynolds number for the transition from laminar to turbulent flow is $\mathrm{Re_{cr}} \sim 30{,}000$; for a smooth golf ball, it's $\mathrm{Re_{cr}} \sim 300{,}000$.

What about the conductive heat flux? It is found empirically that heat flows from hot regions to cold regions, with the flux proportional to the temperature gradient. Mathematically, this is expressed by Fourier's Law:

$$\vec{F} = -K \vec{\nabla} T \ , \tag{2.33}$$

where K is the **coefficient of thermal conductivity**. For a neutral gas,

$$K = \frac{5}{2} c_V \mu \sim \frac{k}{\sigma}\left(\frac{kT}{m}\right)^{1/2} \ . \tag{2.34}$$

Thus, for a gas of neutral atomic hydrogen,

$$K = 2 \times 10^6 \,\mathrm{erg\,cm^{-1}\,s^{-1}\,K^{-1}} \left(\frac{T}{10^4\,\mathrm{K}}\right)^{1/2} . \tag{2.35}$$

Under many astrophysical circumstances, the conductive heat flux is small compared to other energy fluxes. For instance, in stellar interiors, the radiative heat flux and the convective heat flux battle for supremacy, with the conductive heat flux trailing badly in third place. However, in the Sun's transparent corona, to mention an example we'll be discussing in Chapter 11, conduction is the main transport mechanism for heat.

Summary of Results So Far

The conservation equations are

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{u} \tag{2.36}$$

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla}P + \frac{1}{\rho}\vec{\nabla}\cdot\overleftrightarrow{\pi} + \vec{g} \tag{2.37}$$

$$\frac{D\varepsilon}{Dt} = -\frac{P}{\rho}\vec{\nabla}\cdot\vec{u} - \frac{1}{\rho}\vec{\nabla}\cdot\vec{F} + \frac{1}{\rho}\Psi + \frac{1}{\rho}(\Gamma - \Lambda) , \tag{2.38}$$

in conjunction with the equation of state for an ideal monatomic gas

$$\varepsilon = \frac{3P}{2\rho} = \frac{3kT}{2m} . \tag{2.39}$$

In the simplest approximation, we ignore the viscosity and heat conduction by setting $\overleftrightarrow{\pi}$, $\vec{F}$, and Ψ equal to zero. When this approximation is made, the conservation equations are known as the Euler equations.

When viscosity and heat conduction are not negligible, we make the approximations

$$\pi_{ij} = \mu D_{ij} + \beta\vec{\nabla}\cdot\vec{u}\delta_{ij} \tag{2.40}$$

and

$$\vec{F} = -K\vec{\nabla}T . \tag{2.41}$$

In this approximation, the viscous force per unit volume is

$$\vec{\nabla}\cdot\overleftrightarrow{\pi} = \mu\nabla^2\vec{u} + (\beta + \mu/3)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) . \tag{2.42}$$

The rate of viscous energy dissipation is equal to

$$\Psi = \frac{\mu}{2} | \overleftrightarrow{D} |^2 + \beta(\vec{\nabla} \cdot \vec{u})^2 \ , \tag{2.43}$$

where the square of the scalar norm of the deformation tensor is

$$| \overleftrightarrow{D} |^2 = \sum_{i,j} D_{ij} D_{ij} \ . \tag{2.44}$$

Chapter 3

Introduction to Sound & Shocks

Suppose we are in an inviscid, non-heat-conducting, nonradiative medium. We will first consider a case with plane parallel symmetry (all the properties of the gas depend only on x and t). The mass continuity equation is then

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \ . \tag{3.1}$$

The momentum equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g \ . \tag{3.2}$$

If the medium is self-gravitating, the acceleration g is given by Poisson's equation:

$$\frac{\partial g}{\partial x} = -4\pi G\rho \ . \tag{3.3}$$

A uniform static medium, with $\rho = \rho_0$, $u = 0$, and $P = P_0$, will satisfy the equations of continuity and of motion if we perpetuate the **Jeans swindle**. In a homogeneous, isotropic universe, the gravitational acceleration must be $\vec{g}_0 = 0$, by symmetry. However, Poisson's equation will yield $g_0 = 0$ only if $\rho_0 = 0$. From the Newtonian point of view, in other words, an infinite, static, matter-filled universe cannot exist. The Jeans swindle deals with this difficulty by ignoring it.[1] Let us, like Jeans, assume that $g_0 = 0$ for the uniform medium.

[1] If you strongly object to being swindled, try reading "Mathematical Vindications of the Jeans Swindle", by M. Kiessling (astro-ph/9910247).

Now, let us introduce small perturbations to the system, so that

$$\begin{aligned}
\rho &= \rho_0 + \rho_1(x,t) && (3.4)\\
u &= u_1(x,t) && (3.5)\\
P &= P_0 + P_1(x,t) && (3.6)\\
g &= g_1(x,t)\ . && (3.7)
\end{aligned}$$

where $|\rho_1|/\rho_0 \ll 1$ and $|P_1|/P_0 \ll 1$. The linearized perturbation equations are then

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} &= 0 && (3.8)\\
\rho_0 \frac{\partial u_1}{\partial t} + \frac{dP}{d\rho}\Big|_0 \frac{\partial \rho_1}{\partial x} &= g_1 \rho_0 && (3.9)\\
\frac{\partial g_1}{\partial x} &= -4\pi G \rho_1\ . && (3.10)
\end{aligned}$$

In writing the equation of motion, I have made the implicit assumption that $P = P(\rho)$. Taking the time derivative of equation (3.8), and subtracting the spatial derivative of equation (3.9), we find

$$\frac{\partial^2 \rho_1}{\partial t^2} - \frac{dP}{d\rho}\Big|_0 \frac{\partial^2 \rho_1}{\partial x^2} = 4\pi G \rho_0 \rho_1\ . \tag{3.11}$$

If the self-gravitation term on the right hand side is small enough to be ignored, this is just a wave equation with a propagation speed

$$a = \left(\frac{dP}{d\rho}\right)^{1/2}\ . \tag{3.12}$$

The density and pressure perturbations that propagate through the medium with velocity $\pm a$ are **sound waves**. For a polytrope with adiabatic index γ, the sound speed at density ρ_0 and pressure P_0 is

$$a_0 = \left(\frac{\gamma P_0}{\rho_0}\right)^{1/2} = \left(\frac{\gamma k}{m} T\right)^{1/2}\ . \tag{3.13}$$

For a neutral atomic gas, the sound speed is $a = 0.12\,\mathrm{km\,s^{-1}}\,\mu_a^{-1/2}(T/1\,\mathrm{K})^{1/2}$, where μ_a is the mean atomic mass in units of the proton mass.

When is the self-gravity of the medium negligible? Consider a sound wave of the form $\rho_1(x,t) \propto \exp[i(\omega t - kx)]$. In that case, equation (3.11), including the self-gravity term on the right hand side, reduces to the dispersion relation

$$\omega^2 = k^2 a_0^2 - 4\pi G \rho_0 \ . \tag{3.14}$$

Thus, ω is real for wavenumbers $k > k_J$, where

$$k_J \equiv \sqrt{4\pi G \rho_0}/a_0 \ . \tag{3.15}$$

For $k < k_J$, the frequency ω is imaginary, and the perturbations grow exponentially due to their self-gravity.

The **Jeans length** for a neutral atomic gas is

$$\lambda_J \equiv \frac{2\pi}{k_J} \sim 20\,\mathrm{pc}\ \mu_a^{-1} \left(\frac{T}{1\,\mathrm{K}}\right)^{1/2} \left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)^{-1/2} \tag{3.16}$$

and the **Jeans mass** is

$$m_J \equiv \frac{\pi}{6}\lambda_J^3 \rho_0 \sim 100\,\mathrm{M}_\odot\ \mu_a^{-2} \left(\frac{T}{1\,\mathrm{K}}\right)^{3/2} \left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)^{-1/2} \ . \tag{3.17}$$

Sound waves with a wavelength longer than λ_J will collapse gravitationally. On the other end of the size spectrum, sound waves with a wavelength shorter than the mean free path $\lambda = 1/(n\sigma)$ cannot be created. Even in relatively dense molecular clouds, $\lambda \sim 0.2\,\mathrm{AU}$; a sound with this wavelength will have a frequency $\sim 10^{-8}\,\mathrm{Hz}$. From now on, we will deal with sound whose wavelength lies between the mean free path λ and the Jeans length λ_J, so that we can ignore the effects of discreteness and self-gravitation. But what about the effects of viscosity and heat conduction?

The linearized 1-d continuity equation, as always, is

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial u_1}{\partial x} \ . \tag{3.18}$$

The momentum equation, including viscosity, is

$$\rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial x} + \mu' \frac{\partial^2 u_1}{\partial x^2} \ , \tag{3.19}$$

where the effective coefficient of viscosity in the one-dimensional case is $\mu' \equiv 4\mu/3 + \beta$. The energy equation, including heat conduction, is

$$\rho_0 \frac{\partial \varepsilon_1}{\partial t} = -P_0 \frac{\partial u_1}{\partial x} + K \frac{\partial^2 T_1}{\partial x^2} \ . \tag{3.20}$$

The temperature perturbation, in terms of the density and pressure, is

$$T_1 = \frac{m}{k\rho_0}[P_1 - \frac{P_0}{\rho_0}\rho_1] \ . \tag{3.21}$$

The perturbation to the specific internal energy is

$$\varepsilon_1 = \frac{1}{(\gamma-1)\rho_0}[P_1 - \frac{P_0}{\rho_0}\rho_1] \ . \tag{3.22}$$

Using these relations in conjunction with the continuity equation, the energy equation takes the form

$$\frac{\partial}{\partial t}(P_1 - a_0^2\rho_1) = \gamma\chi\frac{\partial^2}{\partial x^2}(P_1 - a_0^2\rho_1/\gamma) \ , \tag{3.23}$$

where $\chi = K/(\rho_0 c_P)$.

If the perturbations are sine waves, of the form

$$\begin{aligned} \rho_1 &= R\exp[i(\omega t - kx)] & (3.24)\\ P_1 &= P\exp[i(\omega t - kx)] & (3.25)\\ u_1 &= U\exp[i(\omega t - kx)] \ , & (3.26) \end{aligned}$$

then the conservation equations yield the relations

$$\begin{aligned} i\omega R - i\rho_0 kU &= 0 & (3.27)\\ -ikP + [i\omega\rho_0 + \mu' k^2]U &= 0 & (3.28)\\ (i\omega + \gamma k^2\chi)P - (i\omega + k^2\chi)a_0^2 R &= 0 \ . & (3.29) \end{aligned}$$

This set of equations yields the dispersion relation

$$\omega^2 = \frac{1 - ik^2\chi/\omega}{1 - i\gamma k^2\chi/\omega}a_0^2k^2 + i\mu' k^2\omega/\rho_0 \ . \tag{3.30}$$

In the absence of viscosity and heat conduction, the dispersion relation is $\omega^2 = a_0^2k^2$. Suppose, however, that we add a small μ and K, so that the wavenumber is now $k = \omega/a_0 + k_1$, with $|k_1| \ll \omega/a_0$. The perturbation to the wavenumber is then

$$k_1 = -i\frac{\omega^2}{2a_0^3\rho_0}[\mu' + (\gamma-1)\rho_0\chi] \ . \tag{3.31}$$

The density perturbation is now

$$\rho_1 = R \exp[i(\omega t - \omega x/a_0)] \exp(-x/L_1) \ , \tag{3.32}$$

with the attenuation length

$$L_1 = \frac{2a_0^3 \rho_0}{\omega^2} [\mu' + (\gamma - 1)\rho_0 \chi]^{-1} \ . \tag{3.33}$$

The viscosity and heat conduction damp the sound waves by converting the sound energy into random kinetic energy.

If we use the sound speed $a^2 \sim kT/m$ and viscosity $\mu \sim (mkT)^{1/2}/\sigma$ for an atomic gas, we find that the attenuation length is

$$L_1 \sim \frac{a_0^2}{\omega^2} n\sigma \sim \frac{\Lambda^2}{\lambda} \ , \tag{3.34}$$

where Λ is the wavelength of the propagating sound and λ is the mean free path of the gas. For sound to propagate, we require $\Lambda \gg \lambda$, and hence $L_1 \gg \Lambda$; the sound is not attenuated significantly until it has traveled for many wavelengths. In air at room temperature, $L_1 \sim 600\,\mathrm{km}(\omega/1000\,\mathrm{Hz})^{-2}$. In the interstellar medium (ISM), $L_1 \sim 700\,\mathrm{AU}(\omega/10^{-10}\,\mathrm{Hz})^{-2}$.

So far, we have been assuming that the sound waves are of infinitesimal magnitude, with $|\rho_1|/\rho_0 \ll 1$, $|P_1|/P_0 \ll 1$, and $u_1 \ll a_0$. As a consequence, we have assumed that the sound speed in the medium has the uniform value $a = a_0$. However, the sound speed is a function of density; for a polytrope, $a \propto \rho^{(\gamma-1)/2}$. As a consequence, crests of sound waves will travel more rapidly than troughs of sound waves, as illustrated in Figure 3.1. Although a wave on the surface of water can be triple-valued, as shown at time 3 in Figure 3.1, creating a "breaker", this is forbidden for a sound wave. Sound waves, therefore, will steepen until a **shock** forms. A shock front is a surface that marks a sudden jump in the density, pressure, and velocity of a gas. The shock front is supersonic – that is, it propagates at a velocity faster than the sound speed in the unshocked medium.

Shocks are ubiquitous in the ISM. Whenever the bulk velocity u is larger than the sound speed, you are likely to form shocks. For instance, there are shocks associated with:

- cloud - cloud collisions ($u \sim 10\,\mathrm{km\,s^{-1}}$),
- galaxy - galaxy collisions ($u \sim 300\,\mathrm{km\,s^{-1}}$),

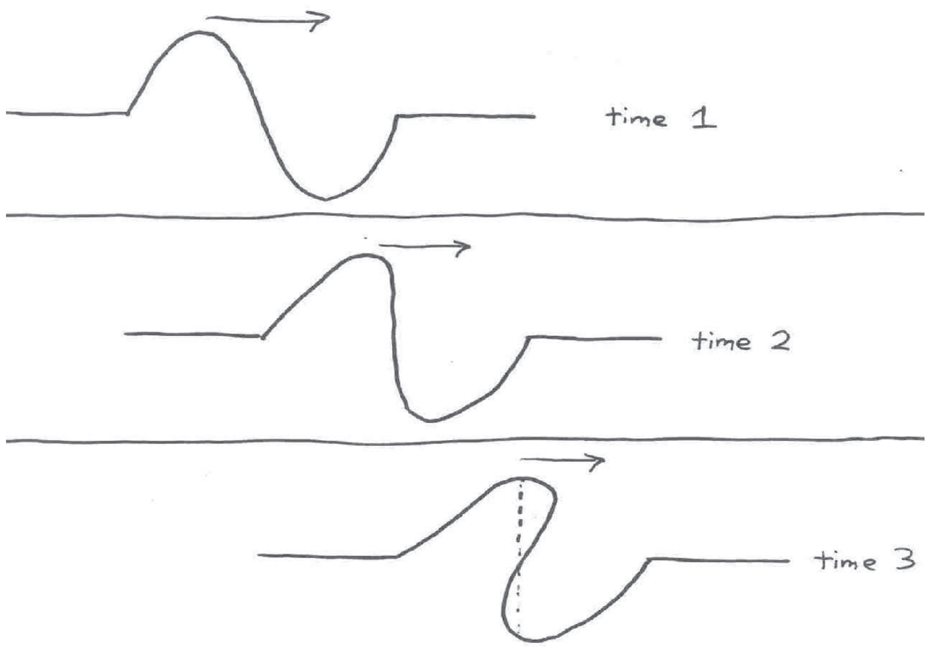

Figure 3.1: The steepening of a sound wave as a crest overtakes a trough.

- stellar winds encountering the ambient ISM ($u \sim 3000\,\mathrm{km\,s^{-1}}$),
- jets from radio galaxies encountering the ambient intergalactic medium ($u \sim 5000\,\mathrm{km\,s^{-1}}$),
- supernova ejecta encountering the ambient ISM ($u \sim 2 \times 10^4\,\mathrm{km\,s^{-1}}$), and
- gas accreting onto neutron stars ($u \sim 10^5\,\mathrm{km\,s^{-1}}$).

Shocks play an important role role in determining the structure of the ISM. For instance, they heat the ISM. Moreover, in the spiral arms of galaxies, shocks compress the gas, which can trigger star formation.

Let us see how shocks behave once they are created through the steepening of a sound wave. Consider, to begin with, a simple plane parallel shock, as shown in Figure 3.2. The math is easiest if we place ourselves in a frame of reference that is comoving with the shock front. Let u_1 be the bulk velocity of the unshocked gas (upstream of the shock) in this frame of reference, and let ρ_1, P_1, and a_1 be the density, pressure, and sound speed of the unshocked gas, which is assumed to be uniform. The bulk velocity, density, pressure, and sound speed immediately downstream from the shock are u_2, ρ_2, P_2, and a_2. The transition layer between the unshocked gas and the postshock gas is very thin. For a neutral atomic gas, $\Delta x \sim \lambda$, the mean free path length. For most purposes, it is adequate to regard the shock transition layer as an infinitesimally thin surface. For a steady-state shock, the conservation equations have the form

$$\begin{aligned} \frac{d}{dx}(\rho u) &= 0 && (3.35)\\ \frac{d}{dx}(\rho u^2 + P) &= 0 && (3.36)\\ \frac{d}{dx}(\rho[u^2/2 + \varepsilon] + P) &= 0\ . && (3.37) \end{aligned}$$

The gas properties immediately before and after being shocked are consequently linked by the **Rankine-Hugoniot jump conditions**:

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 && (3.38)\\ \rho_1 u_1^2 + P_1 &= \rho_2 u_2^2 + P_2 && (3.39)\\ \frac{1}{2}u_1^2 + \varepsilon_1 + P_1/\rho_1 &= \frac{1}{2}u_2^2 + \varepsilon_2 + P_2/\rho_2\ . && (3.40) \end{aligned}$$

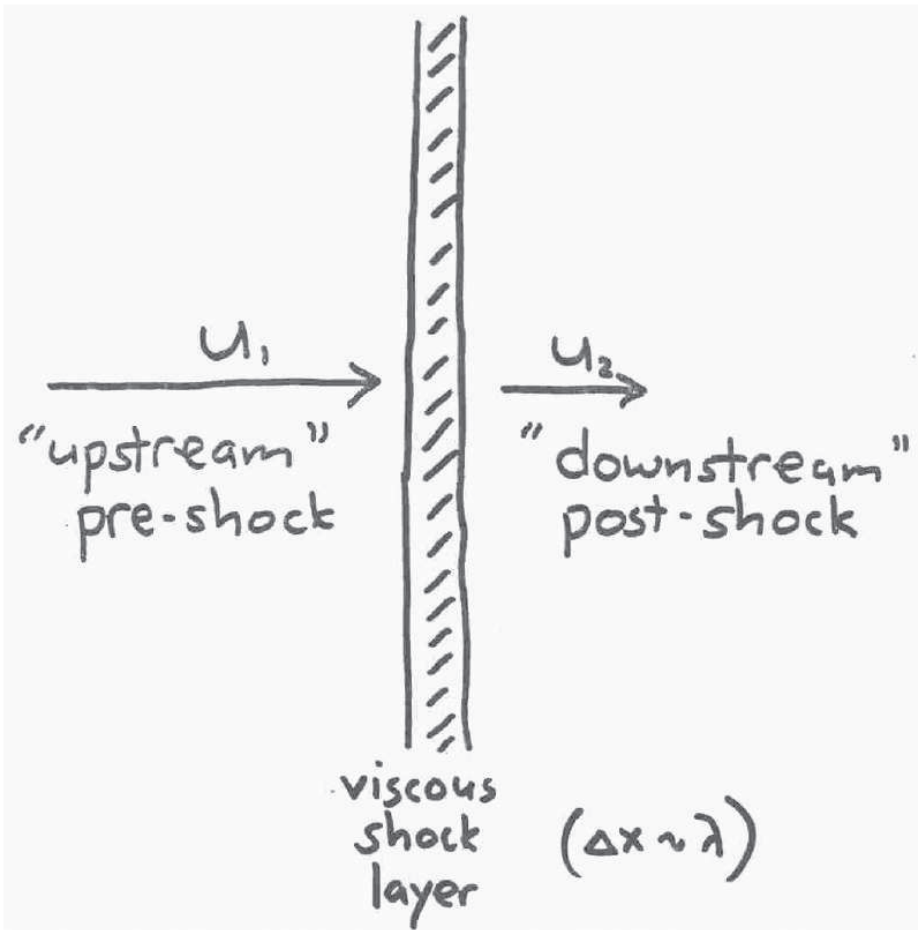

Figure 3.2: The geometry of a plane parallel shock; we are in the shock's frame of reference, with pre-shocked gas flowing in from the left.

For a polytropic gas, the last of the Rankine-Hugoniot conditions may be rewritten in the form

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma - 1}\frac{P_2}{\rho_2} \ . \tag{3.41}$$

I have made the implicit assumption that γ is the same for the preshock and postshock gas.

The Rankine-Hugoniot conditions are just the conservation equations in a new guise. A dimensionless number that is often cited in the context of shocks is the **Mach number**,

$$M_1 \equiv u_1/a_1 = \left(\frac{\rho_1 u_1^2}{\gamma P_1}\right)^{1/2} \ . \tag{3.42}$$

The Mach number is the ratio of the velocity of the shock (relative to the unshocked medium) to the sound speed in the unshocked medium. Using the Rankine-Hugoniot conditions, we may solve for the density, pressure, and temperature jumps in terms of M_1.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} = \frac{u_1}{u_2} \tag{3.43}$$

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} \tag{3.44}$$

$$\frac{T_2}{T_1} = \frac{[(\gamma-1)M_1^2+2][2\gamma M_1^2-(\gamma-1)]}{(\gamma+1)^2 M_1^2} \ . \tag{3.45}$$

A strong shock is defined as one that is highly supersonic, with $M_1 \gg 1$. For a strong shock,

$$\frac{\rho_2}{\rho_1} \approx \frac{\gamma+1}{\gamma-1} \tag{3.46}$$

$$P_2 \approx \frac{2}{\gamma+1}\rho_1 u_1^2 \tag{3.47}$$

$$T_2 \approx \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{m}{k}u_1^2 \ . \tag{3.48}$$

Thus, no matter how strong the shock is, the ratio ρ_2/ρ_1 has a finite value; for a monatomic gas, with $\gamma = 5/3$, the ratio is $\rho_2/\rho_1 = 4$. However, a strong shock is very efficient at converting the bulk kinetic energy of the upstream gas ($\sim \rho_1 u_1^2$) into thermal energy.

A weak shock is defined as one that is barely supersonic, with $M_1 - 1 = \epsilon \ll 1$. For a weak shock,

$$\frac{\rho_2}{\rho_1} \approx 1 + \frac{4}{\gamma+1}\epsilon \tag{3.49}$$

$$\frac{P_2}{P_1} \approx 1 + \frac{4\gamma}{\gamma+1}\epsilon \tag{3.50}$$

$$\frac{T_2}{T_1} \approx 1 + \frac{4(\gamma-1)}{\gamma+1}\epsilon \; . \tag{3.51}$$

Generally speaking, a shock converts supersonic gas ($M_1 > 1$) into subsonic gas (in the shock's frame of reference). It *increases* density, *decreases* bulk velocity (relative to the shock front), *increases* pressure, and *increases* temperature.

The conversion of bulk kinetic energy to random thermal energy occurs by dissipation within the shock layer itself. Within the shock layer, a jump in velocity of $\Delta u \sim a_1 \sim (kT/m)^{1/2}$ occurs over a length $\Delta x \sim \lambda \sim 1/(n\sigma)$. The xx component of the viscous stress tensor within the shock is

$$\pi_{xx} = (\frac{4}{3}\pi + \beta)\frac{\Delta u}{\Delta x} \sim \frac{\sqrt{mkT}}{\sigma}\frac{\sqrt{kT}}{\sqrt{m}}(n\sigma) \sim nkT \; . \tag{3.52}$$

The increase in the specific entropy as the gas crosses the shock front is $s_2 - s_1 = c_P \ln(T_2/T_1) - (k/m)\ln(P_2/P_1)$. For an extremely strong shock ($M_1 \to \infty$), the entropy increase is $s_2 - s_1 \propto \ln M_1$.

So far, we've been dealing with **normal shocks**; that is, shocks in which the the velocity vector $\vec{u}_1$ of the upstream, unshocked gas is perpendicular to the shock front. However, some shocks are better described as **oblique shocks**, in which $\vec{u}_1$ approaches the shock front at an angle other than 90 degrees. For instance, the shock waves associated with spiral arms of galaxies can be approximated as oblique shocks. An oblique shock is illustrated in Figure 3.3. Again, let us put ourselves into a frame of reference that is moving along with a plane parallel shock front. This time, however, the unshocked gas is flowing into the shock at an angle ϕ with respect to the plane of the shock front. Thus, the bulk velocity u_1 can be broken down into a perpendicular component

$$u_{\perp 1} = u_1 \sin\phi \tag{3.53}$$

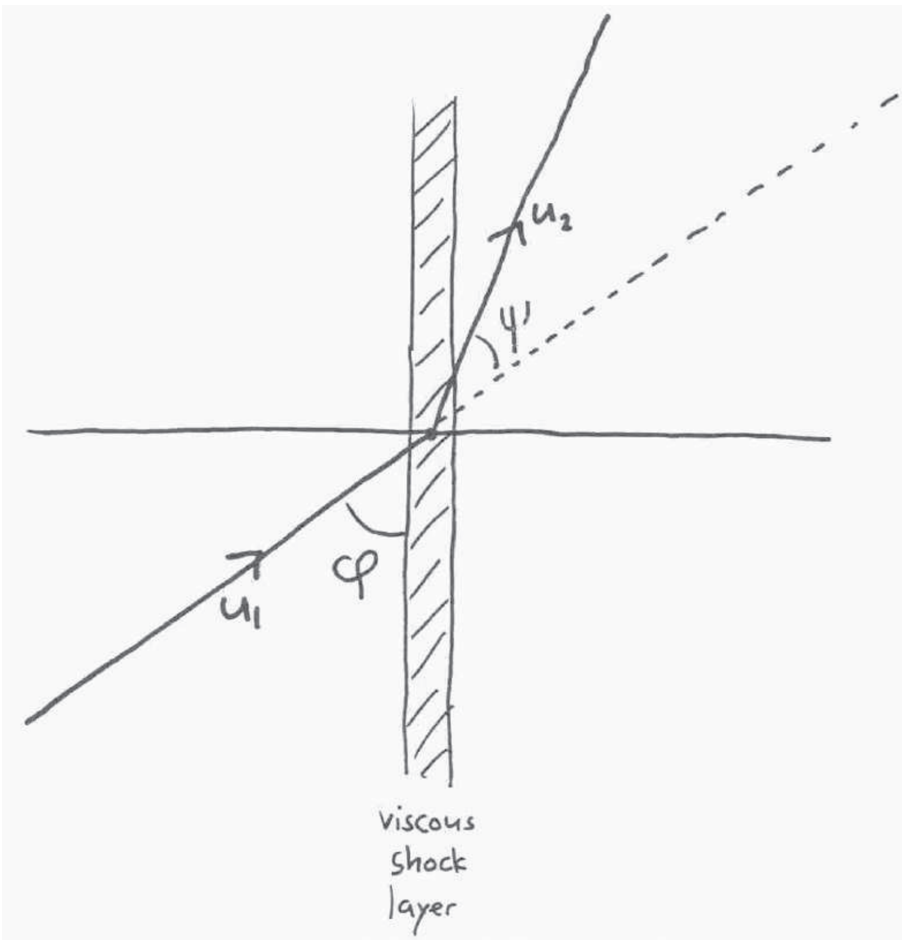

Figure 3.3: The geometry of an oblique shock; we are in the shock's frame of reference, with pre-shocked gas flowing in from the left.

and a parallel component

$$u_{\parallel 1} = u_1 \cos\phi \ , \tag{3.54}$$

which are perpendicular and parallel, respectively, to the shock front. The postshock velocity $\vec{u}_2$ is rotated through an angle ψ away from $\vec{u}_1$ and towards the shock front. The perpendicular and parallel components of $\vec{u}_2$ are

$$u_{\perp 2} = u_2 \sin(\phi - \psi) \tag{3.55}$$

and

$$u_{\parallel 2} = u_2 \cos(\phi - \psi) \ . \tag{3.56}$$

The parallel component of the velocity is unchanged by passage through the shock. Thus,

$$u_1 \cos\phi = u_2 \cos(\phi - \psi) \ . \tag{3.57}$$

The Rankine-Hugoniot jump conditions for the perpendicular component are

$$\rho_1 u_{\perp 1} = \rho_2 u_{\perp 2} \tag{3.58}$$

$$\rho_1 u_{\perp 1}^2 = \rho_2 u_{\perp 2}^2 \tag{3.59}$$

$$\frac{1}{2} u_{\perp 1}^2 + \varepsilon_1 + \frac{P_1}{\rho_1} = \frac{1}{2} u_{\perp 2}^2 + \varepsilon_2 + \frac{P_2}{\rho_2} \ . \tag{3.60}$$

In an oblique shock, $u_\perp$ obeys the same relations as u does in a normal shock. From our previous encounter with the Rankine-Hugoniot jump conditions, we know that the ratio of the perpendicular velocities is

$$\frac{u_{\perp 2}}{u_{\perp 1}} = \frac{2 + (\gamma - 1) M_1^2 \sin^2\phi}{(\gamma + 1) M_1^2 \sin^2\phi} \ , \tag{3.61}$$

where $M_1 = u_1/a_1$ is the Mach number of the upstream, unshocked flow.

Since the parallel component of $\vec{u}$ is conserved and the perpendicular component is decreased, the velocity vector is refracted away from the normal to the shock front. Computing the actual value of the angle ψ through which it is refracted cumbersome in the general case. Combining the equations for the perpendicular and parallel components of the velocity, we find

$$\tan\psi = \frac{1 - \cos 2\phi - 2/M_1^2}{\gamma - \cos 2\phi + 2/M_1^2} \frac{\sin 2\phi}{1 - \cos 2\phi} \ . \tag{3.62}$$

In the case of a strong shock, the relation between ϕ and ψ takes the simpler form

$$\tan\psi = \frac{2\tan\phi}{(\gamma + 1) + (\gamma - 1)\tan^2\phi} \ . \tag{3.63}$$

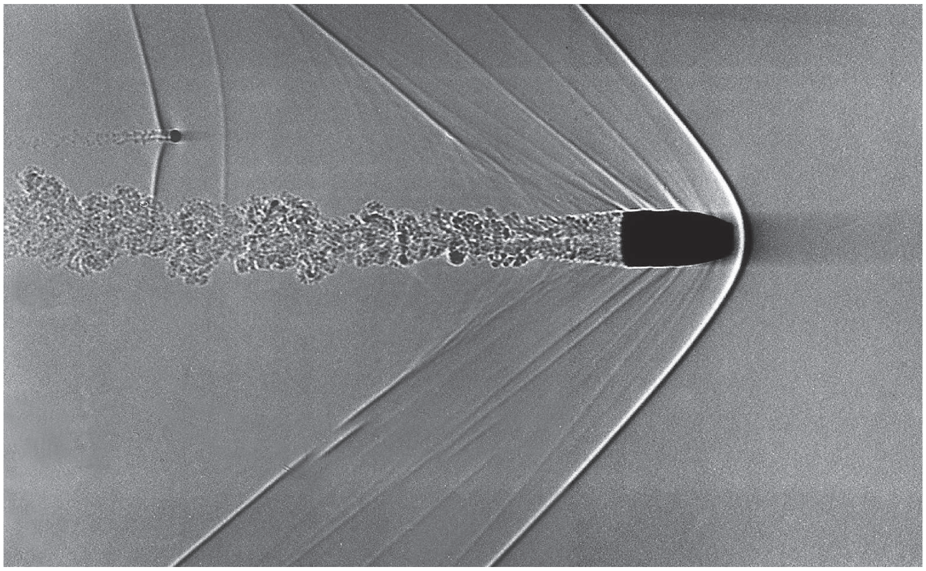

Figure 3.4: Shadowgraph of the detached bow shock in front of a bullet traveling at $M_1 = 1.5$; also note the turbulent wake. (Andrew Davidhazy: RIT).

The maximum value $\psi_{\rm m}$, in this case, is given by the relation

$$\tan\psi_{\rm m} = \left(\frac{1}{\gamma^2 - 1}\right)^{1/2} , \tag{3.64}$$

which occurs when $\tan^2\phi = (\gamma+1)/(\gamma-1)$. For a monatomic gas ($\gamma = 5/3$) the maximum angle of refraction is $\psi_{\rm m} = 36.9°$, which occurs when $\phi = 63.4°$. Even an arbitrarily strong shock can't divert the flow through an angle of 90°. If a blunt object, therefore, moves through a gas at supersonic speeds, then a *detached* bow shock must form ahead of the object, as shown in Figure 3.4.