Manipulating Equations

In derivations in class and on the homework assignments, we often need to start with one set of equations and try to end up with another equation, which tells us the quantity we are trying to find or describes some new physical principle. Here are some reminders on the basics of manipulating algebraic equations.

If we start with an equation $a = b$, we can add the same thing to both sides: $a + c = b + c$. Furthermore, if we have a second equation $c = d$, then $a + c = b + d$, since we are still adding the same thing to both sides.

If we have an equation $a = b$, we can multiply both sides of the equation by the same quantity or divide both sides of the equation by the same quantity:

$$ac = bc \qquad \text{or} \qquad a/c = b/c.$$

The one thing you have to be careful of here is not to divide by zero. (I will use ac and $a \times c$ interchangeably to mean "a times c". $a/c = \frac{a}{c}$ means "a divided by c.")

We can multiply fractions together, and we can reverse the order of multiplications ($ab = ba$), so, for example

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = \frac{a}{d} \times \frac{c}{b}.$$

Often this kind of rearrangement will allow you to divide numbers that are more convenient to divide (e.g., because they have the same units).

Dividing by a fraction is the same as multiplying by the reciprocal of that fraction (where you switch the numerator and denominator), for example

$$\frac{a/b}{c/d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

If you have the same quantity in the numerator and denominator, you can cancel it out

$$\frac{ab}{bc} = \frac{a}{c} \qquad \text{and} \qquad a \times \frac{b}{a} = b.$$

You can take the square root of both sides of an equation, or you can raise both sides of an equation to the same power. If $a = b$, then

$$\sqrt{a} = \sqrt{b}, \qquad a^2 = b^2, \qquad a^{3/2} = b^{3/2}.$$

Also

$$\sqrt{a \times b \times c} = \sqrt{a} \times \sqrt{b} \times \sqrt{c},$$

and

$$(a \times b \times c)^2 = (a \times b)^2 \times c^2 = a^2 \times b^2 \times c^2.$$

As an example, suppose we have the equation $x^2/b = a^2$, and we would like to solve for x. Multiplying both sides of the equation by b gives $x^2 = a^2 b$. Taking the square-root of both sides gives $x = a\sqrt{b}$, since the square-root of x^2 is x and the square-root of a^2 is a.

OVER

Physical quantities usually come with units, and these units multiply and divide and get raised to powers along with the numbers attached to them. For example, one meter divided by one second is 1 m/sec; one watt divided by one square-meter is 1 watt/m^2.

If you divide two numbers with the same units, then the units cancel out leaving you with a pure number, e.g., (20 m / 10 m) = 2. Often you can make your life easier by setting up ratios so that units cancel out in this way as much as possible.

Sometimes a homework question will ask you to produce an equation as an answer, and sometimes it will ask you to produce a number with units (e.g., the distance to the Andromeda galaxy, or the value of Hubble's constant). In the second case, you want to first get an equation for the quantity you are trying to find, then substitute values to get your number. It is always easiest to work with symbols (f, L, d, r, θ, etc.) for as long as you can, and only substitute numbers at the end; otherwise you do a lot of unnecessary work multiplying numbers and it is easy to make a mistake along the way.

With these rules in mind, let's investigate a couple of applications of our first equation, $f = L/4\pi d^2$.

(1) The luminosity of the sun is 3.9×10^{26} watts, and the flux of the sun at the earth's surface is 1380 watts/m^2. What is the distance from the earth to the sun, in meters?

Since we want a distance, let's first solve the equation for distance by multiplying both sides by d^2, dividing by f, and taking the square root of both sides.

$$f = \frac{L}{4\pi d^2} \implies d^2 = \frac{L}{4\pi f} \implies d = \sqrt{\frac{L}{4\pi f}}\ .$$

Now substitute values:

$$d = \sqrt{\frac{3.9 \times 10^{26}\,\text{watts}}{4\pi \times 1380\,\text{watts/m}^2}} = 1.5 \times 10^{11}\text{m}.$$

Note that the watts in the numerator cancelled the watts in the denominator, that 1/m^2 in the denominator becomes m^2 in the numerator (i.e., dividing by 1/m^2 is the same as multiplying by m^2), and that taking the square root of m^2 yields m, so our answer has the desired units.

(2) Star B has twice the intrinsic luminosity of star A, but it is twice as far away. What is its apparent flux relative to that of star A?

You may be able to answer this question in your head: the apparent flux is proportional to the luminosity and to the inverse square of the distance, so star B appears $2/2^2 = 1/2$ as bright as star A.

To do it more formally with the equation, note that we are being asked for the ratio f_B/f_A. We know that

$$f_B = L_B/4\pi d_B^2 \qquad \text{and} \qquad f_A = L_A/4\pi d_A^2,$$

because our flux-luminosity-distance equation should apply to both stars individually. To get our desired ratio, divide the left side of the first equation by the left side of the second equation and the right side of the first equation by the right side of the second equation:

$$\frac{f_B}{f_A} = \frac{L_B/4\pi d_B^2}{L_A/4\pi d_A^2} = \frac{L_B}{L_A} \times \frac{4\pi d_A^2}{4\pi d_B^2} = 2 \times \frac{1}{4} = \frac{1}{2}.$$

This strategy — write an equation twice and take ratios — is often useful, and it often allows you to cancel out numerical constants (like 4π) or physical constants (like c or G) before you ever have to plug them into your calculator.