5. Newtonian Gravity

Reading: The reading for this section is Chapter 4. It is really only section 4.1 that we are covering in this section of the course, but it is useful to read ahead to sections 4.2 and 4.3 to see where we will be going.

The Big Bang Theory

If we imagine "running Hubble flow backwards," we conclude that all matter in the observable universe was densely packed together at a finite time in the past.

However, there are ways of escaping this conclusion.

My preferred definition of the big bang theory is this: The universe has expanded from a very hot, very dense state that existed at a finite time in the past (approximately 14 billion years ago).

This hypothesis makes empirically testable predictions, which turn out to describe our universe remarkably well.

This formulation does not say much about the nature of the "big bang" itself, or anything about why it occurred.

Those are interesting questions, and towards the end of the course we will look at current ideas about them if we have time.

Two Theories of Gravity

To make further progress, we need to understand gravity and its influence on the universe.

In physics, we usually talk about two distinct theories of gravity and how it works: Isaac Newton's and Albert Einstein's.

For many purposes, Newton's theory, which he described in the 1687 book *Principia Mathematica*, is highly accurate.

Einstein's theory, known as General Relativity and completed in 1915, is conceptually very different. However, it yields nearly identical predictions to Newton's theory for objects moving much slower than the speed of light, provided that gravity is "weak" (as it is whenever one is far from the event horizon of black holes).

When the two theories differ in their predictions, Einstein's theory is clearly more accurate, giving perfect agreement with a wide variety of high precision experiments.

Einstein's theory is needed to fully understand the universe. However, we will use Newton's theory where we can, because it is much easier to apply.

Velocity

Velocity is the rate of change of position.

For an object moving in a straight line at fixed speed: $v = d/t$

Equivalently, $d = vt$.

But velocity has a magnitude *and* a direction. In mathematical terms, represented by a vector (roughly speaking, an arrow).

The magnitude of the velocity (length of the vector) is called the speed.

Objects moving with the same speed in different directions have different velocities.

Acceleration

Acceleration is the rate of change of velocity.

It is also a vector, with a magnitude and a direction.

If you start from rest, the direction of the acceleration tells which direction you will go, and the magnitude tells how rapidly you will reach a certain speed.

For motion in a straight line, starting from rest, with constant acceleration for time t,

velocity at beginning = 0
velocity at end = at
average velocity = $\frac{1}{2}at$
distance traveled = (avg. velocity) $\times t = \frac{1}{2}at \times t = \frac{1}{2}at^2$.
Note that

$$d = \frac{1}{2}at^2$$

is the formula for the distance d traveled in time t by an object that *starts from rest* and moves with *constant acceleration* a.

For example, the acceleration of falling objects near the surface of the Earth is 9.8 m/sec^2.
Note that m/sec^2 is a compact way of writing "meters-per-second-per-second". If the acceleration is a m/sec^2, then in 1 second the velocity will change by a m/sec.

If an object starts at rest and falls for one second, its velocity is $v = at = (9.8\,\mathrm{m/sec}^2) \times (1\mathrm{s}) = 9.8\,\mathrm{m/sec}$.

The distance it travels is $d = \frac{1}{2}at^2 = \frac{1}{2} \times (9.8\,\mathrm{m/sec}^2) \times (1\mathrm{s})^2 = 4.9\,\mathrm{m}$.

Newton's Laws of Motion

In his 1687 book *Principia Mathematica*, Newton summarizes his theory of motion and forces with three "laws":

1. The principle of inertia: a body remains at rest, or moves in a straight line at constant speed, unless acted upon by a net outside force.

2. Force on a body produces acceleration that is proportional to the strength of the force and inversely proportional to the mass of the body.

3. When one body exerts a force on a second body, the second body exerts an equal and opposite force on the first body.

While Newton presented these as empirically discovered "laws" of nature, they really constitute a *theory* of motion and forces that is highly successful in explaining a wide range of phenomena.

F = ma

For our purposes, we care mostly about Newton's second law, which is best summarized by a fundamental equation:

$$F = ma$$

F = force [kg-m/s^2, a unit that is also called a newton]
m = mass [kg]
a = acceleration [m/s^2]
Note that this is equivalent to $a = F/m$, which is more obviously a translation of the words to an equation.
Note also that no force implies no acceleration, so the second law, $F = ma$, automatically incorporates the first law, the principle of inertial.

The *mass* of a body is, effectively, its "amount of stuff." It can be measured, from Newton's second law, by measuring the body's resistance to acceleration. The standard, metric system unit of mass is the kilogram (kg).

F and a are really vectors, with a magnitude and a direction.

m is just a number, with no direction.

If we're being careful, therefore, we write $\vec{F} = m\vec{a}$ to convey that $\vec{F}$ and $\vec{a}$ are vectors and m is not.

Newton's Law of Gravity
Newton proposes the following: There is a force of gravitational attraction between any two bodies that is proportional to the product of their masses and inversely proportional to the square of the distance between their centers.
In the form of an equation:

$$F = \frac{GMm}{r^2}$$

F = gravitational force
M = mass of first body
m = mass of second body
r = distance between their centers
G = Gravitational Force Constant (a.k.a. Newton's constant)

The direction of the force on one body is always towards the other body (attractive).
Double the mass of one body, and the force doubles.
Double the mass of both bodies, and the force goes up by four.
Double the distance between the bodies, and force goes down by four.
Halve the distance between the bodies, and force goes up by four.

In metric units, Newton's constant is measured to be $G = 6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$.
However, we will often formulate problems in a way that gets G to cancel out.

Weight
The *weight* of a body is the force that gravity exerts on it.
For an object at the earth's surface, the weight of an object of mass m is

$$F = \frac{GM_{\mathrm{earth}}m}{R^2_{\mathrm{earth}}}.$$

(For reference, $M_{\mathrm{earth}} = 6 \times 10^{24}$ kg and $R_{\mathrm{earth}} = 6.4 \times 10^6$ m.)
The weight is therefore proportional to the mass m, so in everyday use we often don't distinguish the two.
For example, we often think of a kg as a measure of weight, but it is really a measure of mass.
A 1 kg ball on, say, the Moon, would have the same mass, but it would weigh less (1/6 as much).
A 1 kg ball at an altitude of 6.4×10^6 m above the earth's surface would have the same mass, but it would weigh 1/4 as much as it does on the earth's surface because it is a distance $2R_{\mathrm{earth}}$ from the earth's center.

Notions of Mass
Newton's equations give us two different ways to measure mass.
$F = ma$ suggests that we measure mass as a resistance to acceleration.
$F = GMm/r^2$ suggests that we measure mass as an ability to exert gravitational force.

Galileo was the first to demonstrate clearly that, on the earth, falling objects of different mass have the same acceleration, provided one can ignore air resistance (which is a non-gravitational effect).
According to Newton's equations, the *acceleration* of mass m under the influence of gravity,

$$a = \frac{F}{m} = \frac{GM}{r^2},$$

does not depend on m!

This prediction agrees with Galileo's finding, and it is more general. For example, an asteroid and a planet will follow the same orbit around the sun if they are at the same distance, even though the planet is much more massive than the asteroid.

The gravitational acceleration of an object does not depend on its mass, because the extra force on a more massive object is canceled out by its greater resistance to acceleration.

In Newton's theory, this cancellation looks like a coincidence, but Einstein takes it as a central clue in developing his theory of gravity two centuries later.

All material objects that we encounter in everyday life are made of atoms.

Atoms are made of protons, neutrons, and electrons.

Protons and neutrons have nearly equal mass, while electrons are 2000 times less massive.

For a "normal" object made of atoms, therefore, the mass is essentially a count of how many protons and neutrons it contains.

Circular Motion

Going in a circle at constant speed requires acceleration.

Q: Why?

A: Because the *direction* of the velocity is changing, even though the magnitude (speed) is not.

An object moving in a circle of radius r at constant speed v has acceleration

$$a = \frac{v^2}{r}$$

a = acceleration
v = speed of motion
r = radius of circle

We still usually write v in this formula, even though it is only the magnitude of the velocity (the speed) that is constant.

The direction of the acceleration is *toward the center of the circle.* Note that this direction is constantly changing.

Measuring Mass with Orbiting Objects

Consider an object of mass m orbiting in a circular orbit of radius r under the gravitational influence of a more massive body of mass M.

For example, m could be the mass of a planet and M could be the mass of the sun, or m could be the mass of a moon or satellite and M could be the mass of a planet.

The acceleration in a circular orbit is $a = v^2/r$, in the direction of the center of the circle.

The acceleration provided by gravity is $a = GM/r^2$.

Setting these two equal, we get

$$\frac{v^2}{r} = \frac{GM}{r^2},$$

which is an equation we can solve for M to get

$$M = \frac{v^2 r}{G}.$$

v = speed of orbiting object

r = radius of (circular) orbit
G = Newton's gravitational constant
M = mass of object that is *being* orbited (i.e., producing the gravity that is causing the mass m to orbit)
If we measure an object's orbital radius and orbital velocity, we can figure out the mass of the body it is orbiting around.

Examples
The orbital velocity of the earth is 30,000 m/sec (or 30 km/sec), which you can compute knowing the circumference of the orbit (2π AU) and the period (1 year).
The radius of the earth's orbit is 1.5×10^{11} AU.
Using the value of G I gave earlier, you can compute the mass of the sun:

$$M_{\rm sun} = \frac{(30,000\,{\rm m/s})^2 (1.5 \times 10^{11}\,{\rm m})}{6.67 \times 10^{-11} \frac{\rm m^3}{{\rm kg}-sec^2}} = 2.0 \times 10^{30}\,{\rm kg}.$$

Jupiter has an orbital radius of 5.2 AU and a period of 11.8 years, making its orbital speed 13,100 m/sec.
If you do the same calculation using the numbers for Jupiter, you will get the same mass of the sun. Good!

Newton couldn't get the mass of the sun this way because the value of G itself wasn't measured in the laboratory until 1798. The value of the AU was also not well known.
However, Newton could get the mass of Jupiter relative to the mass of the sun by using the orbit of Jupiter's moons.
For example, Jupiter's largest moon, Ganymede, orbits Jupiter once every 7.2 days, and by measuring angles and motions in the solar system you can find that the distance from Ganymede to Jupiter is 7.15×10^{-3} AU, even if you don't know how many meters that corresponds to.
The orbital speed of the earth can be written $v_{\rm earth} = 2\pi \times (1\,{\rm AU})/365\,{\rm days}$.
The orbital speed of Ganymede can be written $v_{\rm gany} = 2\pi \times (7.5 \times 10^{-3}\,{\rm AU})/7.2\,{\rm days}$.
We know that

$$M_{\rm sun} = \frac{v_{\rm earth}^2 r_{\rm earth}}{G} \qquad ; \qquad M_{\rm Jup} = \frac{v_{\rm gany}^2 r_{\rm gany}}{G}.$$

Take the ratio of the left sides and the right sides to get

$$\frac{M_{\rm Jup}}{M_{\rm sun}} = \left(\frac{v_{\rm Gany}}{v_{\rm earth}}\right)^2 \left(\frac{r_{\rm Gany}}{r_{\rm earth}}\right) = \left(\frac{2\pi \times (7.5 \times 10^{-3}\,{\rm AU})/7.2\,{\rm day}}{2\pi \times (1\,{\rm AU})/365\,{\rm day}}\right)^2 \left(\frac{7.5 \times 10^{-3}\,{\rm AU}}{1\,{\rm AU}}\right) = 0.001.$$

The mass of Jupiter is 1000 times less than the mass of the sun.

Successes of Newton's Theory
Newton's critical innovations are:

- To provide a general account of the relation between force, mass, and acceleration, encapsulated by the formula $F = ma$
- To propose that gravity is a *universal* phenomenon: anything with mass generates a gravitational force, and anything with mass is affected by gravitational force.
- To propose a quantitative description of the gravitational force: $F = GMm/r^2$

He shows that this theory of forces, motion, and gravity explains a wide range of observed phenomena.

Newton's theory

- Explains why all bodies fall at the same rate near the earth's surface, independent of mass
- Explains why the acceleration of the moon, at a distance of 60 earth radii, is $1/60^2 = 1/3600$ the acceleration of objects at earth's surface. (The acceleration can be found from $a = v^2/r$ using the moon's distance and orbital period.)
- Explains the three basic empirical facts about the motion of planets known as Kepler's laws:

    1. Planets follow elliptical orbits with the sun at one focus of the ellipse.

    2. A planet moves faster when it is closer to the sun, with the sun-planet line sweeping out equal areas in equal times.

    3. The square of a planet's orbital period around the sun is proportional to the cube of its distance from the sun. For circular orbits, this is equivalent to the statment that $v \propto 1/\sqrt{r}$, where v is the orbital speed and r is the distance from the sun. (And $\propto$ means "proportional to.")

- Explains why the moons of Jupiter *also* obey Kepler's 3rd law, but with a different constant of proportionality because they are orbiting Jupiter, whose mass is smaller than the mass of the sun.
- Explains the basic phenomena of ocean tides. Because the moon pulls more strongly on one side of the earth than the other, it stretches the earth (mainly the oceans) into an ellipsoid, pointing towards the moon. Newton's theory explains why there are two high tides a day, when the moon is directly overhead and directly underfoot.
- Predicts that comets move on highly elongated elliptical orbits, turning rapidly when they get close to the sun. Newton's theory correctly explains the observed motions of comets.
- Predicts that planets should have a (small) gravitational effect on each other. When Jupiter is "catching up" to Saturn, Saturn should slow down, then speed up after Jupiter passes. This effect had already been seen, but never understood, and Newton's laws explain it precisely. This is smoking gun evidence for universal gravity.

In 1781, the planet Uranus was discovered serendipitously during a telescopic survey of the sky. Fifty years later, it was showing deviations from the expected orbit, and two mathematicians show these deviations could be caused by the gravity of a more distant planet. The planet Neptune was found within 1 degree of the predicted position.

With modern observational techniques, tests of Newtonian gravity have become far more detailed and precise.

It explains a wide range of phenomena, though in fact it turns out to be slightly inaccurate. These inaccuracies are explained by Einstein's theory of gravity, General Relativity.

Unification

By providing a unified explanation of terrestrial and celestial phenomena, Newton establishes the essential principle that underlies all of modern astronomy: the basic laws of physics apply *throughout the universe.*

We use physics learned in laboratories on earth to understand planets, stars, galaxies, and the universe.

We use astronomical observations to learn about physics beyond the reach of terrestrial experiments.