3. Universal Gravity

Basic Concept

Newton's second law: force causes acceleration.
Implication: dropped objects fall to Earth because they are pulled by a force.

Newton's idea: The *same* force causes the Moon to orbit the Earth.
But the force weakens with distance from the center of the Earth, so the Moon falls more slowly.
The *same* force, exerted by the Sun, causes planets to orbit the Sun.

The Moon's "fall" is its deviation from a straight path.
In one second, the Moon falls 0.00136 meters (about 1.4 mm).
An object on the Earth (e.g., an apple), falls 4.9 meters in 1 second.
The distance to the Moon is about 60 Earth radii.
(4.9 meters)/(0.00136 meters) = 3600.

What Newton Had To Explain

The basic empirical facts known by the time Newton came up with universal gravity were:

- The acceleration at the earth's surface is 9.8 m/sec^2. Gravity accelerates all objects at the same rate, independent of mass.
- Kepler's laws of planetary motion:

  1. Planets move on elliptical orbits, with the sun at one focus.
  2. The sun-planet line sweeps out equal areas in equal times (so a planet moves faster when it is closer to the sun).
  3. The square of a planet's orbital period is proportional to the cube of the orbit's semi-major axis:

  $$(P/1\,\text{year})^2 = (r_p/1\,\text{AU})^3,$$

  where r_p is the planet's semi-major axis and 1 AU = 1 Astronomical Unit is the earth-sun distance (precisely, the semi-major axis of the earth's orbit).
  (Note that r_p is conventionally denoted as a, but we are going to use a for acceleration.)
- The moon moves on an elliptical orbit with the earth at its focus and obeys a similar equal-area rule. The moon's acceleration is about 1/3600 the acceleration at earth's surface.
- The orbits of the four large moons of Jupiter (the only ones known at the time) also follow Kepler's third law, but with a different constant of proportionality.

Rewriting Kepler's 3rd Law

The orbits of planets around the Sun are approximately circular.
For now, ignore the small departures from circularity.
Let m_p denote the mass of a planet.
Let r_p denote the radius of the orbit.
Let v_p denote the speed of the planet in its orbit.
The orbital period P is just the circumference $2\pi r_p$ of the orbit divided by the speed v_p, $P = 2\pi r_p/v_p$.

For convenience, let's introduce

$$\begin{aligned}
r_E &= \text{radius of Earth's orbit} = 1\,\text{AU} = 150\,\text{million km} \\
P_E &= \text{period of Earth's orbit} = 1\text{yr} = 3.1\times 10^7\text{s} \\
v_E &= \frac{2\pi r_e}{P_E} = \text{Earth's orbital speed} = 30\,\text{km}\,\text{s}^{-1}
\end{aligned}$$

Now we'll rewrite Kepler's 3rd law, for the case of circular orbits:

$$\begin{aligned}
\left(\frac{r_p}{r_E}\right)^3 &= \left(\frac{P}{P_E}\right)^2 \\
&= \left(\frac{2\pi r_p}{v_p} \times \frac{v_E}{2\pi r_E}\right)^2 \\
&= \left(\frac{v_E}{v_p}\right)^2 \times \left(\frac{r_p}{r_E}\right)^2
\end{aligned}$$

Dividing both sides of this equation by $(r_p/r_E)^2$, we get

$$\left(\frac{r_p}{r_E}\right) = \left(\frac{v_E}{v_p}\right)^2,$$

which we can also write in the form

$$v_p = v_E \times \sqrt{\frac{r_E}{r_p}}.$$

For example, Saturn, with an orbital radius of 9 AU, has an orbital speed of $30\,\mathrm{km\,s^{-1}}/\sqrt{9} = 10\,\mathrm{km\,s^{-1}}$.

The Inverse-Square Law

Recall that the acceleration in a circular orbit at constant speed is $a = v^2/r$.
Combined with our rewritten form of Kepler's 3rd law, this implies

$$\begin{aligned}
a_p &= \frac{v_p^2}{r_p} \\
&= \left(v_E^2 \times \frac{r_E}{r_p}\right) \times \frac{1}{r_p} \\
&= \left(v_E^2 \times \frac{r_E^2}{r_E}\right) \times \frac{1}{r_p^2} \\
&= \frac{v_E^2}{r_E} \times \left(\frac{r_E}{r_p}\right)^2 \\
&= a_E \times \left(\frac{r_E}{r_p}\right)^2 \quad \propto \frac{1}{r_p^2},
\end{aligned}$$

where $\propto$ in the last line means "proportional to."
What force is needed to keep planets in circular orbits obeying Kepler's 3rd law?

$$F = ma \propto \frac{m_p}{r_p^2}.$$

The force should be *proportional to the planet's mass* and *proportional to the inverse square of the planet's distance from the Sun.*

Newton's Law of Gravity

Newton proposes the following: There is a force of gravitational attraction between any two bodies that is proportional to the product of their masses and inversely proportional to the square of the distance between their centers.

In the form of an equation:

$$F = \frac{GMm}{r^2}$$

F = gravitational force
M = mass of first body
m = mass of second body
r = distance between their centers
G = Gravitational Force Constant (a.k.a. Newton's constant)

Double the mass of one body, and the force doubles.
Double the mass of both bodies, and the force goes up by four.
Double the distance between the bodies, and force goes down by four.
Halve the distance between the bodies, and force goes up by four.

Two Comments

Note that the *acceleration* of mass m,

$$a = \frac{F}{m} = \frac{GM}{r^2},$$

does not depend on m!

The gravitational acceleration of an object does not depend on its mass, because the extra force on a more massive object is canceled out by its greater resistance to acceleration.

This is a deep fact that will play a crucial role in leading Einstein to his theory of gravity two centuries later.

Newton posits the existence of the gravitational force between separated bodies.
He does not try to explain a "cause" for it.
Some scientists of the time considered this "action at a distance" to be a fatal weakness of Newton's theory.

Measuring Mass with Orbiting Objects

Consider an object of mass m orbiting in a circular orbit of radius r under the gravitational influence of a more massive body of mass M.

For example, m could be the mass of a planet and M could be the mass of the Sun, or m could be the mass of a moon or satellite and M could be the mass of a planet.

The acceleration in a circular orbit is $a = v^2/r$, in the direction of the center of the circle.
The acceleration provided by gravity is $a = GM/r^2$.
Setting these two equal, we get

$$\frac{v^2}{r} = \frac{GM}{r^2},$$

which is an equation we can solve for M to get

$$M = \frac{v^2 r}{G}.$$

If we measure an object's orbital radius and orbital velocity, we can figure out the mass of the body it is orbiting around.

To get a mass in kg, we need to know the value of G, which was not measured until the late 1700s. Even without knowing the value of G, one can get ratios of masses by dividing in a way that makes G cancel out; for example, Newton was able to figure out the ratio of Jupiter's mass to the Sun's mass (as you will do in Homework 1).

We will usually set things up so that we can divide G away, but if you want to know its value you can always get it if you remember that the Schwarzschild radius for one solar mass is

$$\frac{2GM_\odot}{c^2} = 3\,\mathrm{km} = 3000\,\mathrm{m}.$$

If we substitute $1M_\odot = 2 \times 10^{30}$ kg and $c = 3 \times 10^8$ m/s, then solve for G, we get

$$G = \frac{c^2 \times 3000\,\mathrm{m}}{2M_\odot} = \frac{(3 \times 10^8\,\mathrm{m/s})^2 \times 3000\,\mathrm{m}}{4 \times 10^{30}\,\mathrm{kg}} = 6.75 \times 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}.$$

(I have rounded some numbers here, and a more accurate value is 6.67×10^{-11} instead of 6.75×10^{-11}.)

Successes of Newton's Theory

Newton's theory

- Explains why all bodies fall at the same rate near the earth's surface, independent of mass
- Explains why the acceleration of the moon, at a distance of 60 earth radii, is $1/60^2 = 1/3600$ the acceleration of objects at earth's surface
- Predicts that planets moving in circular orbits about the Sun will obey Kepler's 3rd law.

Newton demonstrates that his theory explains all three of Kepler's laws, i.e., that planets moving under the influence of an inverse-square law from the sun will

- follow elliptical orbits with the sun at one focus,
- sweep out equal areas in equal times,
- obey Kepler's 3rd law.

The same arguments apply to the moons of Jupiter, except that constant of proportionality in Kepler's 3rd law is smaller because the mass of Jupiter is smaller than the mass of the sun.

Thus, Newton's theory explains all of the basic empirical facts listed above, and it provides a way to measure the mass of Jupiter relative to the mass of the sun.

Other successes:

- Because the moon pulls more strongly on one side of the earth than the other, it stretches the earth (mainly the oceans) into an ellipsoid, pointing towards the moon. Newton's theory explains why there are two high tides a day, when the moon is directly overhead and directly underfoot.
- Newton predicts that comets move on highly elongated elliptical orbits, turning rapidly when they get close to the sun. His theory correctly explains the observed motions of comets.
- Planets should have a (small) gravitational effect on each other. When Jupiter is "catching up" to Saturn, Saturn should slow down, then speed up after Jupiter passes. This effect had already been seen, but never understood, and Newton's laws explain it precisely. This is smoking gun evidence for universal gravity.
- The planet Uranus is discovered serendipitously in 1781 during a telescopic survey of the sky. Fifty years later, it is showing deviations from the expected orbit, and two mathematicians show these deviations could be caused by the gravity of a more distant planet. The planet Neptune is found within 1 degree of the predicted position.

By providing a unified explanation of terrestrial and celestial phenomena, Newton establishes the essential principle that underlies all of modern astronomy: the basic laws of physics apply *throughout the universe.*

We use physics learned in laboratories on earth to understand planets, stars, galaxies, and the universe.

We use astronomical observations to learn about physics beyond the reach of terrestrial experiments.

What allowed the Newtonian Revolution to happen?

Trade leading to increasing focus on time, technology, mathematics in Europe.

Reformation and secular nobility undermining authority of the Church.

New confidence in humanity's ability to understand nature.

Improvements in astronomical observation techniques: precise positional measurements, telescopes.

Specific achievements of Copernicus, Brahe, Kepler, Galileo — Newton had the problem "set up" for him to solve.

Active scientific communities in England and in continental Europe.

Development of necessary mathematical tools by Newton himself.