Astronomy 142: Assignment 3

This assignment is due at the beginning of class on *Friday*, Feb 20. You may consult with others in the class when you are working on the homework, but (a) you should make a first attempt at everything on your own before talking to others, and (b) you must write up your eventual answers independently.

You are welcome to come to my office hours or David Nataf's office hours for advice (remember that you can make an appointment if need be). Office hours are listed on the syllabus.

Please be sure that your name is on your assignment, and please staple or paper clip all sheets together.

Assignments that are late will be marked down 10 points (out of 100) if they are in my mailbox by 5 pm Friday, or by 20 points if they are in my mailbox by 5 pm Monday. Assignments later than that will not be accepted unless you have made prior arrangements with me.

Part I: Short Questions (Each worth 5 points)

1. My former friend dropped me from a hovering rocket ship straight towards a black hole. Before I enter the event horizon and disappear, how could I tell whether the black hole I am falling into is spinning or not spinning?

2. The black hole at the center of the Milky Way has a mass about 4 million times the mass of the sun. What is the radius of its event horizon, in km?

3. The quantum "exclusion principle" tells us that if electrons are squeezed into a small volume, then most of the electrons must move with high momentum. Why is this result crucial to the existence of white dwarf stars?

4. What is the source of energy that powers a supernova explosion?

5. What did Oppenheimer and Volkoff demonstrate that was crucial to making the (theoretical) case that black holes should exist?

Part II: Degeneracy Pressure (Each part worth 5 points)

Consider a white dwarf of mass $1M_\odot$ and radius $R = 6000$ km (about the radius of the earth). The white dwarf is supported by the pressure of degenerate electrons. The mass of an electron is 1/2000 the mass of a proton, and in a white dwarf there is 1 electron for every proton and 1 neutron for every proton (the neutron mass is almost identical to the proton mass). Therefore, the mass of the electrons in the white dwarf is $M_{\text{electrons}} = M_\odot/4000$.

(a) Using our approximate formula for pressure balance in a star, $E \sim GM^2/R$, show that

$$E_{\text{electrons}} \sim M_{\text{electrons}} c^2.$$

(Remember that the Schwarzschild radius for $1M_\odot$ is 3 km.)

(b) What is the connection between this result and the Chandrasekhar mass limit for white dwarfs? (Recall that the kinetic energy of an individual electron of mass m is $\frac{1}{2}mv^2$, at least when $v \ll c$.)

(c) Instead consider a neutron star, still $1M_\odot$ but with a radius of $R = 20$ km. Now the source of pressure is degenerate *neutron* pressure instead of degenerate electron pressure, and it is the energy of the neutrons that matters. Also, the star is made entirely of neutrons, which are 2000 times more massive than electrons, so $M_{\text{neutrons}} = M_\odot$. Show that

$$E_{\text{neutrons}} \sim M_{\text{neutrons}} c^2.$$

Part III: Bending of Light (Each part worth 5 points)

As we discussed in class, light passing near a star (or planet, or other massive body) will be bent by the star's gravity. You can calculate the amount of bending approximately (within a factor of 4 or so) using the same "steel box" example that we used to discuss the bending of light in class. The diagram on the attached page will be useful for this purpose. It shows a light ray passing at a distance r from a star of mass M. We can make the approximation that the light ray is not affected by star's gravity when it is much further than r (because gravity falls off as $1/r^2$), and that the light ray changes direction while it passes through an imaginary "box" that is $2r$ on a side and whose center is a distance r away from the star. The dashed line shows the path that the light ray would follow if it were not deflected by the star's gravity.

(a) What is the gravitational acceleration g at the center of the box? (Give your answer as a formula relating g to G, M, and r.)

(b) How long does it take the light ray to cross the box? (To answer this part, just assume that the light travels on a straight path; the actual bending will typically be much smaller than I have drawn it in the figure.)

(c) Using the equivalence principle as discussed in class, argue that by the time the light has crossed the box it has "fallen" by an amount $l = \frac{2GM}{c^2}$. (You should derive this equation from other equations, including the one in part (a), and you should also have a few words explaining why this is the right calculation according to the equivalence principle.)

(d) Expressed in radians, the angle θ by which the light path bends is (approximately) l/r, the drop divided by half the size of the box. Show that

$$\theta = \frac{R_{\mathrm{Sch}}}{r},$$

where $R_{\mathrm{Sch}} = \frac{2GM}{c^2}$ is the Schwarzschild radius for a black hole of mass M.

(e) Suppose that the mass in question is the sun, and that you are observing a background star along a line of sight that almost grazes the edge of the sun. In this case, $r = R_{\odot} = 7 \times 10^5$ km. Recalling that the Schwarzschild radius for $M = 1M_{\odot}$ is 3 km, compute the angle of light bending produced by the sun's gravity. Convert your answer from radians to degrees using the fact that there are 57.3 degrees per radian.

(More exactly, there are $360/2\pi$ degrees per radian, since there are 2π radians and 360 degrees in a circle.)

(f) As you can see from (e), a normal star like the sun can only produce small light deflections, much less than one degree. A black hole can produce much stronger light deflection (e.g., tens of degrees or more). Why?

(Refer to what you have done in parts (d) and (e). It is not sufficient to just say that the black hole has "stronger gravity.")