3. The Friedmann-Robertson-Walker Metric

Reading: Chapter 3.4-3.6

The Einstein Model

In spherical coordinates, the Minkowski metric (no spacetime curvature, special relativity) can be written

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2,$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

is the angular separation.

In 1917, Einstein introduced the first modern cosmological model, based on GR, in which the *spatial* metric is that of a 3-sphere:

$$dl^2 = dr^2 + R^2\sin^2(r/R)d\Omega^2.$$

Here R is the curvature radius of the 3-dimensional space, and r is distance from the origin.

In the coordinate frame of a freely falling observer, time is just proper time as measured by the observer, and the spacetime metric is

$$ds^2 = -c^2dt^2 + dl^2.$$

This spacetime metric describes a homogeneous, isotropic, and static (unchanging) universe.

Remarkably, Einstein found that such a universe was not a mathematically valid solution of his Field Equation.

He responded by changing the equation, adding a "cosmological term" to allow the existence of a static, homogeneous universe.

Although he didn't think of it this way, Einstein's modification was equivalent to adding a form of energy with constant density and negative pressure.

In GR, this form of energy has repulsive gravity, which balances the attractive gravity of the matter in the universe and allows it to remain static. (However, the solution is unstable.)

The FRW Metric

A natural generalization of the Einstein model is to allow the curvature radius to be a function of time.

The universe is still homogeneous and isotropic on a surface of constant t, but it is no longer static.

In the 1930s, Robertson and Walker (independently) showed that there are only three possible spacetime metrics for a universe that is homogeneous and isotropic.

They can be written

$$ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + S_k^2(r)d\Omega^2\right],$$

where

$$\begin{aligned} S_k(r) &= R_0\sin(r/R_0), \qquad k = +1, \\ &= r, \qquad k = 0, \\ &= R_0\sinh(r/R_0), \qquad k = -1. \end{aligned}$$

In this notation (the same as that used by Ryden), $a(t)$ is dimensionless.

It is defined so that $a(t_0) = 1$ at the time t_0 (usually taken to be the present) when the curvature radius is R_0. At other times the curvature radius is $a(t)R_0$.

The radial coordinate r, the radius of curvature R_0, and $S_k(r)$ all have units of length (e.g., Mpc).

At a time t when the expansion factor is $a(t)$, the spatial metric for $k = 0$ is

$$dl^2 = d(ar)^2 + (ar)^2 d\Omega^2.$$

This resembles the spatial part of the Minkowski metric, but the physical scale associated with a given comoving radius r changes in proportion to $a(t)$.

For $k = +1$, the spatial metric is

$$dl^2 = d(ar)^2 + (aR_0)^2 \sin^2(ar/aR_0) d\Omega^2.$$

This resembles the spatial metric of the static Einstein model, but the physical scale and the curvature radius are proportional to $a(t)$.

For $k = -1$, the spatial metric is

$$dl^2 = d(ar)^2 + (aR_0)^2 \sinh^2(ar/aR_0) d\Omega^2.$$

Remember that

$$\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right).$$

It is instructive to use the Taylor expansions of $\sin x$ and $\sinh x$ to show that the curved cases approach the flat case in the limit $r \ll R_0$.

Friedmann and LeMaître used this metric in their cosmological models of the 1920s. Robertson and Walker proved that they are the only forms consistent with the Cosmological Principle (homogeneity and isotropy).

It is usually called the Friedmann-Robertson-Walker (FRW) metric, or sometimes the Robertson-Walker metric.

Comoving Observers

The metric depends on the coordinate frame of the observer.

Even a homogeneous and isotropic universe only appears so to a special set of freely falling observers, called *comoving observers*.

These observers are “going with the flow” of the expanding universe. They have constant values of r, θ, and ϕ, and the proper distance between them increases in proportion to $a(t)$.

In the coordinate frame of these observers, the FRW metric applies, and the time coordinate of the FRW metric is just proper time as measured by these observers.

An observer moving relative to the local comoving observers has a “peculiar velocity,” where peculiar is used in the sense of “specific to itself” rather than “odd.”

An observer with a non-zero peculiar velocity does not see an isotropic universe – e.g., dipole anisotropy of the cosmic microwave background caused by reflex of the peculiar velocity.

Space curvature vs. spacetime curvature

For $k = +1$, the space geometry at constant time is that of a 3-sphere, positively curved. The total volume of the universe is finite, though it grows in proportion to $a^3(t)$.

For $k = 0$, the space geometry at constant time is Euclidean, a.k.a. "flat space." Space is infinite.

For $k = -1$, the space geometry at constant time is that of a negatively curved, 3-dimensional "pseudo-sphere." Space is infinite.

Note: these are descriptions of *space* at constant t.

For many forms of $a(t)$, *spacetime* is positively curved even if *space* is not, (this is always the case unless a cosmological constant or some other form of energy with negative pressure is important).

The substitution $x = S_k(r)$ allows the FRW metric to be written in another frequently used form:

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dx^2}{1 - kx^2/R_0^2} + x^2d\Omega^2\right].$$

because $du = S_k'(r)dr = (1 - kS_k^2(r))^{1/2}d\omega = (1 - ku^2)^{1/2}dr$ for $k = +1, 0, -1$.

With r as radial coordinate, radial distances are "Euclidean" but angular distances are not (unless $k = 0$). With x as radial coordinate, the reverse is true.

Proper distance and the Hubble parameter

In a curved, expanding universe, the idea of "distance" becomes complex. There are several reasonable and useful definitions of the distance to an object, or the distance between two objects.

The most straightforward of these definitions is the "proper distance" d_p, the distance to an object as measured in a surface of constant time.

If we follow a radial ray ($d\Omega = 0$) to an object with comoving radial coordinate r, the metric tells us that $dl = a(t)dr$ and thus

$$d_p = \int_0^r a(t)dr = a(t)r.$$

The value of r is the *comoving distance* to the object. The comoving distance doesn't change as the universe expands. With our definition that $a(t_0) = 1$, the comoving distance is equal to the proper distance at the present day.

The proper distance changes with time at a rate

$$\dot{d}_p = \dot{a}r = \frac{\dot{a}}{a}d_p.$$

Thus, the expanding universe leads to Hubble's law, $v = Hd$, with the Hubble parameter

$$H = \frac{\dot{d}_p}{d_p} = \frac{\dot{a}}{a}.$$

This is an *important equation*, relating the empirical parameter H discovered by Hubble to the expansion parameter of the Friedmann equation.

I've written H instead of H_0, because this identification of the Hubble parameter with $\dot{a}/a$ holds at any time, not just the present.

Discovery of cosmic expansion, a capsule history

1915: Einstein completes GR.

1917: Einstein proposes homogeneous, static cosmological model.

1923: Edwin Hubble demonstrates that the Andromeda nebula is a distant stellar system, a galaxy.

1920s:

- Vesto Slipher and others measure velocities of spiral nebulae via Doppler shifts. Most are receding.
- Hubble and others estimate distances to spiral nebulae, via Cepheids, other bright stars.

Friedmann and Lemaître develop GR-based cosmological models with an expanding universe.

1929: Hubble publishes "distance-velocity relation," a linear proportionality between distance and velocity: $\mathbf{v} = H_0\mathbf{r}$.

Several recent articles have discussed contributions of Friedmann, Lemaître, and Slipher to the discovery of "Hubble's law."

Redshift of photons

The frequency ν of a photon emitted by a nearby comoving source at distance d is Doppler shifted by a fractional amount

$$\frac{d\nu}{\nu} = \frac{-v}{c} = \frac{-Hd}{c} = -Hdt,$$

where H is the Hubble parameter and the last equality follows because $d = c\,dt$.

Substituting $H = \dot{a}/a$ yields

$$\frac{d\nu}{\nu} = -\frac{\dot{a}\,dt}{a} = -\frac{da}{a} \implies d\ln\nu = -d\ln a.$$

Let the photon be emitted with frequency ν_e at time t_e and observed with frequency ν_o at time t_o.

Integrate to get

$$\frac{\nu_e}{\nu_o} = \frac{a_o}{a_e} = \frac{\lambda_o}{\lambda_e} \equiv (1+z), \qquad z = \text{redshift}.$$

Constant of integration fixed by demanding $\nu_o \longrightarrow \nu_e$ as $a_o \longrightarrow a_e$.

The wavelength of a freely propagating photon stretches in proportion to $a(t)$.

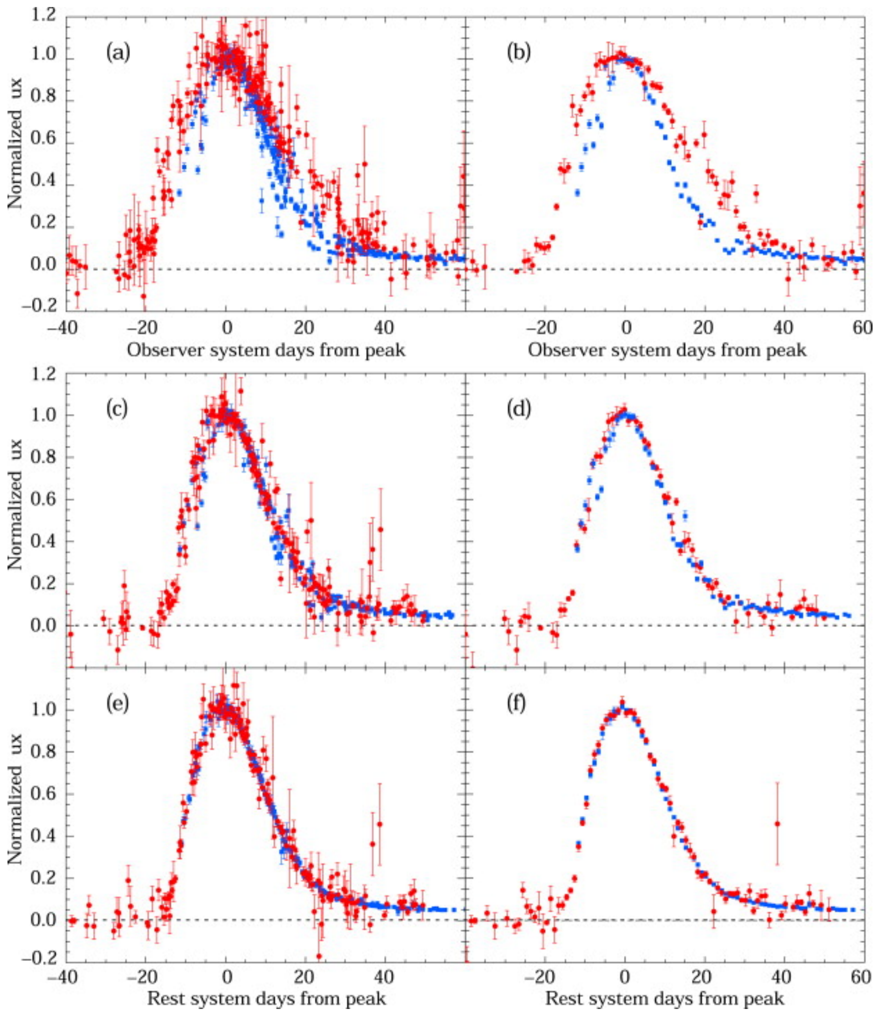
This result can also be derived by considering successive wave crests traveling along null geodesics, as discussed in the textbook, or by solving the GR equation for evolution of 4-momentum along null geodesic.

This result underpins most of observational cosmology: if we can measure the redshift of light from a source, we know the expansion factor $a(t)$ at the time the source emitted its light.

Since light waves don't "disappear," the frequency shift implies time dilation: our clocks run slow compared to those of the emitting object.

This is a real effect. For example, if we observe distant (substantially redshifted) supernovae, they appear to rise and fall more slowly than nearby supernovae.

In the rest frame of the supernova, the rise and fall is the same as it is nearby, but in our frame the time has been stretched.



Time dilation of supernova light curves. Top left shows light curve points from 18 nearby (blue) and 35 high-redshift (red) supernovae. Top right shows these points averaged over 1-day intervals. Panels (c) and (d) show the same after removing the time dilation expected from redshift. From Goldhaber et al. 2001, ApJ, 558, 359.

Comoving distance and redshift

For a radial ray ($d\Omega = 0$), the metric implies

$$c\,dt = a(t)dr,$$

which makes sense because $a(t)dr$ is the physical (proper) distance corresponding to coordinate separation dr at time t.

We can integrate this equation to get the comoving distance r to an object at redshift z:

$$r = \int_{t_e}^{t_0} dr = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)}.$$

The definition $H = \dot{a}/a$ implies $dt = da/(aH)$.

The definition $a = (1+z)^{-1}$ implies $da = -(1+z)^{-2}dz = -a^2 dz$.

Therefore, the comoving distance to a source at redshift z is

$$r = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_{a_e}^{a_0} \frac{c\,da}{a^2(t)H(t)} = \int_0^z \frac{c\,dz}{H(z)} = \frac{c}{H_0}\int_0^z dz \frac{H_0}{H(z)}.$$

To actually evaluate this integral, we need to know the evolution of $H(z)$, which means we need to understand something about the dynamics of the expanding universe.