4. Cosmic Dynamics: The Friedmann Equation

Reading: Chapter 4

Newtonian Derivation of the Friedmann Equation

Consider an isolated sphere of radius R_s and mass M_s, in uniform, isotropic expansion (Hubble flow).
The equation of motion for $R_s(t)$ can be obtained from the gravitational acceleration at the outer edge of the sphere:

$$\frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s^2(t)}.$$

Multiplying both sides by dR_s/dt and integrating converts this "acceleration equation" to an "energy equation":

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U.$$

Mathematically, U is just a constant of integration, but physically it corresponds to the total energy per unit mass at the surface of the expanding sphere, i.e., the sum of the kinetic energy per unit mass and the gravitational potential energy per unit mass.
If $U > 0$, then the expanding sphere has positive total energy and will expand forever (the r.h.s. will always be positive).
If $U < 0$, then the sphere has negative total energy and will eventually recollapse (the r.h.s. will eventually become zero).

Write the radius in the form

$$R_s(t) = a(t)r_s,$$

where r_s is the "comoving" radius of the sphere, equal to the physical radius at the epoch when $a(t) = 1$.
With

$$M_s = \frac{4\pi}{3}\rho(t)R_s^3(t),$$

the energy equation becomes

$$\frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}Gr_s^2\rho(t)a^2(t) + U.$$

Dividing each side by $r_s^2a^2/2$ yields

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2a^2(t)}.$$

Since $\rho(t) \propto 1/a^3(t)$, we see that if U is negative the r.h.s. of this equation will eventually hit zero, after which the expansion reverses.

Although this derivation describes an isolated sphere, Newton's "iron-sphere" theorem tells us that it should also describe any spherical volume of a homogeneous and isotropic universe, since the gravitational effects of a spherically symmetric external matter distribution cancel.
A generalization of the "iron-sphere" theorem turns out to hold in GR as well.

The Friedmann Equation in GR

A proper derivation of the Friedmann equation begins by inserting the Friedmann-Robertson-Walker metric into the Einstein Field Equation.

Since GR yields the Newtonian limit, we should expect the small scale behavior to resemble that of our Newtonian derivation above, and it does, with two important changes.

First, the mass density $\rho(t)$ is replaced by the total energy density $\epsilon(t)/c^2$, which includes rest mass energy and other forms of energy (e.g., energy of photons, or thermal energy of atoms).

[In most texts, this total energy density is just written as $\rho(t)$ and understood to include all contributions, not just rest mass. I will try to stick with Ryden's more pedagogical notation here.]

Second, the "potential energy" term is intimately tied to the curvature of space.

The GR form of the Friedmann Equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2}\frac{1}{a^2(t)},$$

where R_0 is the present value of the curvature radius and $k = +1$, 0, or -1 is the curvature index in the FRW metric.

While the precise form of the last term is not obvious without a GR derivation, it makes reasonable sense that positive space curvature is associated with stronger gravity and thus with negative "binding energy."

If $k \leq 0$, and the energy density is positive, then the r.h.s. is always positive, and an expanding universe continues to expand forever.

If matter is the dominant form of energy, then dilution implies $\epsilon(t) \propto 1/a^3(t)$. If $k = +1$, then the r.h.s. must eventually reach zero, after which the expansion will reverse.

Thus, positive space curvature corresponds to a bound universe.

However, a form of energy for which $\epsilon(t)$ falls more slowly than $1/a^2(t)$, such as a cosmological constant, can change this automatic correspondence.

The Critical Density and the Density Parameter, Ω

Substituting $H(t) = \dot{a}/a$ allows us to write the Friedmann equation in terms of the Hubble parameter,

$$H^2(t) = \frac{8\pi G}{3}\frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2}\frac{1}{a^2(t)}.$$

From this equation, we can see that space is flat ($k = 0$) if the mean density of the universe equals the *critical density*

$$\rho_c(t) = \frac{\epsilon_c(t)}{c^2} = \frac{3H^2(t)}{8\pi G}.$$

The two cosmological equations most worth memorizing are $H = \dot{a}/a$ and this definition of the critical density. Together they *are* the Friedmann equation for a flat universe.

The present day value of the critical density is

$$\rho_{c,0} = \frac{\epsilon_{c,0}}{c^2} = \frac{3H_0^2}{8\pi G}$$
$$= 9.2 \times 10^{-30}\,\mathrm{g\,cm^{-3}} \left(\frac{H_0}{70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}\right)^2$$
$$= 1.4 \times 10^{11} M_\odot\,\mathrm{Mpc^{-3}} \left(\frac{H_0}{70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}\right)^2 .$$

Cosmologists frequently describe the energy density of the universe in terms of the *density parameter*

$$\Omega \equiv \frac{\epsilon}{\epsilon_c} = \frac{\epsilon}{c^2} \times \frac{8\pi G}{3H^2},$$

the ratio of the total energy density ϵ to the critical energy density.
Substituting this definition into the Friedmann equation yields

$$H^2 = \Omega H^2 - \frac{kc^2}{R_0^2 a^2(t)} \quad \Longrightarrow \quad 1 - \Omega(t) = -\frac{kc^2}{H^2(t)a^2(t)R_0^2}.$$

If $\Omega = 1$, then it equals one at all times, since the r.h.s. of this equation always vanishes.
In other cases, the value of Ω changes with time, but if $\Omega > 1$ it is always > 1, and if $\Omega < 1$ it is always < 1, because the r.h.s. cannot change sign.

At the present day, we can solve this equation to get

$$R_0 = \frac{c}{H_0} \left|1 - \Omega_0\right|^{-1/2} .$$

If Ω_0 is very close to one, then the curvature radius is large compared to the Hubble radius c/H_0, and curvature effects on this scale are small.

Evolution of Energy Density: The Fluid Equation

The Friedmann equation determines $a(t)$ if we know H_0 and the energy density $\epsilon(t)$ as a function of time.
(By comparing ϵ_0 to the critical density, we can determine whether $k = +1$, 0, or -1, and we can use our last equation to determine R_0.)
If the only energy contribution is from non-relativistic matter, then $\epsilon(t) = \epsilon_0(a/a_0)^{-3}$, since expansion of the universe simply dilutes the density of particles.

For the more general case, let's turn to the first law of thermodynamics,

$$dE = -P\,dV + dQ,$$

the change of internal energy of a volume of fluid is the sum of $P\,dV$ work and added heat.
The expansion of a homogeneous universe is *adiabatic*, as there is no place for "heat" to come from, and no "friction" to convert energy of bulk motion into random motions of particles.

(There is a caveat to this statement: when particles annihilate, such as electrons and positrons, this adds heat and makes the expansion temporarily non-adiabatic. This matters at some specific epochs in the very early universe.)

Therefore,

$$dE + P\,dV = 0 \implies \dot{E} + P\dot{V} = 0.$$

For a sphere of comoving radius r_s, $V = \frac{4\pi}{3} r_s^3 a^3(t)$,

$$\dot{V} = \frac{4\pi}{3} r_s^3 \left(3a^2\dot{a}\right) = V\left(3\frac{\dot{a}}{a}\right),$$

and $E = V\epsilon$.
Therefore

$$\dot{E} = V\dot{\epsilon} + \dot{V}\epsilon = V\left(\dot{\epsilon} + 3\frac{\dot{a}}{a}\epsilon\right).$$

Together with $\dot{E} + P\dot{V} = 0$, we get

$$V\left(\dot{\epsilon} + 3\frac{\dot{a}}{a}\epsilon + 3\frac{\dot{a}}{a}P\right) = 0$$

and thus

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0.$$

This *fluid equation* describes the evolution of energy density in an expanding universe.

To solve this equation, we need an additional *equation of state* relating P and ϵ.
Suppose we write this in the form

$$P = w\epsilon.$$

In principle, w could change with time, but we will assume that any time derivatives of w are negligible compared to time derivatives of ϵ. This is reasonable if the equation of state is determined by "microphysics" that is not directly tied to the expansion of the universe.
The fluid equation then implies

$$\frac{\dot{\epsilon}}{\epsilon} = -3(1+w)\frac{\dot{a}}{a},$$

with solution

$$\frac{\epsilon}{\epsilon_0} = \left(\frac{a}{a_0}\right)^{-3(1+w)}.$$

For non-relativistic matter,

$$w = P/\epsilon \sim \frac{mv_{\rm th}^2}{mc^2} \sim \frac{v_{\rm th}^2}{c^2} \ll 1,$$

where $v_{\rm th}$ is the thermal velocity of particles.
To a near-perfect approximation, $w = 0$, implying $\epsilon \propto a^{-3}$, in line with our simple dilution argument.

For radiation (i.e., photons), $w = 1/3$, implying $\epsilon \propto a^{-4}$.

This behavior also follows from a simple argument: the number density of photons falls as $n \propto a^{-3}$, and the energy per photon falls as $h\nu \propto a^{-1}$ because of cosmological redshift.

The fluid equation will lead us to some less obvious conclusions when we consider "dark energy."

The Acceleration Equation

If we multiply our standard version of the Friedmann equation by a^2, we get

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon a^2 - \frac{kc^2}{R_0^2}.$$

Take the time derivative

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left(\dot{\epsilon}a^2 + 2\epsilon a\dot{a}\right),$$

divide by $2\dot{a}a$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\epsilon}\frac{a}{\dot{a}} + 2\epsilon\right),$$

and substitute from the fluid equation

$$\dot{\epsilon}\frac{a}{\dot{a}} = -3\left(\epsilon + P\right)$$

to get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\epsilon + 3P\right).$$

We see that if ϵ and P are positive, the expansion of the universe decelerates.

Higher P produces stronger deceleration for given ϵ, e.g., a radiation-dominated universe decelerates faster than a matter-dominated universe.

The appearance of $\epsilon/c^2 + 3P/c^2$ is a specific example of a more general phenomenon in GR: pressure appears in the stress-energy tensor, and it therefore has a gravitational effect. (It must, because the division between energy density and pressure depends on the state of motion of the observer.)

With the stress-energy tensor of an ideal fluid, the Newtonian limit of GR yields a "Poisson equation"

$$\nabla^2\Phi = 4\pi G(\rho + 3P/c^2).$$

The Cosmological Constant, Vacuum Energy, and Cosmic Acceleration

A negative energy density would be pretty bizarre.

Negative pressure sounds bizarre too, but it's not quite as crazy. For a fluid to have negative pressure means that it has *tension* — it takes work to expand the fluid, instead of taking work to compress it.

Suppose the universe is pervaded by a form of energy that is constant density, in space *and* time.

Such an energy could conceivably arise as a consequence of the quantum vacuum, which is not really empty but filled with "virtual" particles.

Expanding a volume increases its energy, so that energy must have come from doing $P\,dV$ work on the volume, which implies that the pressure must be negative (the volume "resists stretching.")

In terms of our fluid equation, we see that $\epsilon =$ const. implies $w = -1$, and thus $P = -\epsilon$.

From the acceleration equation, we see that any fluid with $P < -\epsilon/3$ causes acceleration of the universe, instead of deceleration. Such a fluid has, in GR, repulsive gravity.

This discussion makes cosmological acceleration sound almost reasonable.

The problem is that no one knows how to calculate the quantum vacuum energy from first principles.

In the absence of a complete theory of quantum gravity, the most reasonable guess is that

$$\epsilon_{\rm vac} \sim E_p/l_p^3,$$

where $l_p = (\hbar G/c^3)^{1/2} = 1.6 \times 10^{-35}$ m is the Planck length and $E_p = (\hbar c^5/G)^{1/2} = 2.0 \times 10^9$ J is the Planck energy.

This estimate exceeds the critical density by 120 orders of magnitude!

Since the only "natural" number close to 10^{-120} is zero, the general expectation at least until the mid 1990s was that a "correct" calculation of quantum vacuum energy would produce cancellations that make the answer exactly zero.

It's possible that vacuum energy really does have the value required to produce the observed cosmic acceleration.

It's also possible that the fundamental vacuum energy is zero, and that acceleration of the Universe is caused by some other negative pressure fluid, or by a breakdown of GR on cosmological scales.

Einstein introduced (in 1917) the "cosmological constant" Λ with a different conception, as a modification of the curvature term of the Field Equation rather than an additional contribution to the stress-energy tensor.

However, the gravitational effects of Einstein's cosmological constant are identical to those of a form of energy with $P = -\epsilon$, which remains constant in space and time as the universe expands.

Einstein needed the "repulsive gravity" of the Λ-term to allow a static universe, counteracting the gravitational attraction of matter. He abandoned the idea when the universe was found to be expanding.

But today we need a cosmological constant, or something like it, to explain how the universe can accelerate.