

5. Cosmic Expansion History

Reading: Chapters 5 and 6

Solutions of the Friedmann Equation

We now understand the origin of the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2} \frac{1}{a^2(t)},$$

and the dependence of energy density $\epsilon(t)$ on the expansion factor $a(t)$ for various forms of matter and energy.

Now we want to know what the *solutions* of the Friedmann equation are for interesting cases.

For single-component universes, the following cases can be derived by integrating the Friedmann equation or verified by substitution. You are doing three of them in Problem Set 4.

Matter-dominated, critical density ($k = 0$)

$$\epsilon = \epsilon_0(a/a_0)^{-3}, \quad a(t) = (t/t_0)^{2/3}, \quad H = 2/3t.$$

Radiation-dominated, critical density ($k = 0$)

$$\epsilon = \epsilon_0(a/a_0)^{-4}, \quad a(t) = (t/t_0)^{1/2}, \quad H = 1/2t.$$

Empty ($k = -1$)

$$\epsilon = 0, \quad a(t) = (t/t_0), \quad H = 1/t.$$

Cosmological constant, critical density ($k = 0$)

$$\epsilon = \epsilon_\Lambda = \frac{3H_0^2 c^2}{8\pi G}, \quad a(t) = e^{H_0(t-t_0)}.$$

For the $H(t)$ relation in the matter dominated case, note that if $a \propto t^{2/3}$ then

$$\frac{d \ln a}{d \ln t} = \frac{t}{a} \frac{da}{dt} = tH = \frac{2}{3}$$

and thus $H = 2/3t$. The same argument works for the radiation-dominated and empty cases.

Matter-dominated solutions with curvature are more complicated, but the solutions can be expressed in parametric form (textbook equations 6.17 and 6.18 for $k = +1$, and 6.20 and 6.21 for $k = -1$).

For $k = +1$, the universe expands, reaches a maximum at

$$a_{\max} = \frac{\Omega_0}{\Omega_0 - 1},$$

then recollapses and ends in a “big crunch.”

For $k = -1$, the universe expands forever.

From the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2} \frac{1}{a^2(t)},$$

we can see that the two terms on the right-hand side are equal when $\Omega_0 = 0.5$.

At much earlier times, the first term must dominate, and the universe should evolve like a critical density universe.

At much later times, the second term must dominate, and the universe should evolve like an empty universe.

Specifically, when $a \ll \Omega_0/(1 - \Omega_0)$, the expansion is very close to $a(t) \propto t^{2/3}$.

When $a \gg \Omega_0/(1 - \Omega_0)$, the expansion is very close to $a(t) \propto t$, i.e., “free expansion” (not accelerating or decelerating).

A flat universe with matter and a cosmological constant

A flat universe dominated by matter and a cosmological constant appears to be a good description of the cosmos we live in.

The behavior is analogous to that of the $k = -1$ matter-dominated solution discussed above, except that the transition is to exponential expansion rather than free expansion.

The Friedmann equation in this case can be written

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\Omega_{m,0} \left(\frac{a}{a_0}\right)^{-3} + \Omega_{\Lambda,0} \right].$$

(Since $a_0 \equiv 1$, we could just drop it from this equation.)

The transition occurs at an expansion factor

$$\left(\frac{a_{m\Lambda}}{a_0}\right) \sim \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}.$$

Note that for a critical density universe, we must have $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$.

For a flat universe in which matter and a cosmological constant are the dominant energy components one can integrate the Friedmann equation to get an implicit but analytic solution for the $a(t)$ relation (textbook eq. 6.28):

$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \ln \left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1 + \left(\frac{a}{a_{m\Lambda}}\right)^3} \right].$$

Given an object’s redshift, one can infer the time t_e at which its light was emitted, up to a factor of H_0^{-1} .

Matter-radiation equality and the “benchmark model”

The energy density of radiation in the universe is $\epsilon_{r,0} = a_{\text{SB}} T^4$, where $T = 2.7$ K is the temperature of the cosmic microwave background (CMB).

Roughly, CMB photons have $\lambda \sim 1$ mm, $kT \sim 10^{-3}$ eV, $n_\gamma \sim 10^3$ photons cm^{-3} , implying $\epsilon_{r,0} \sim 1 \text{ eV cm}^{-3}$.

The contribution of starlight is negligible compared to the CMB.

The mean density of hydrogen atoms is roughly one atom per cubic meter. Hence $\epsilon_{\text{bary},0} \sim 10^9 \text{ eV}/10^6 \text{ cm}^3 \sim 10^3 \text{ eV cm}^{-3} \sim 10^3 \epsilon_{r,0}$.

Since $\epsilon_{\text{bary}} = \epsilon_{\text{bary},0}(a_0/a)^3 = \epsilon_{\text{bary},0}(1+z)^3$, and $\epsilon_r = \epsilon_{r,0}(a_0/a)^4 = \epsilon_{r,0}(1+z)^4$, the radiation and baryon densities were equal at

$$(1+z) \sim \frac{\epsilon_{\text{bary},0}}{\epsilon_{r,0}} \sim 1000.$$

To be more precise, we need to use precise numbers for the above *and* include two other important contributions.

The radiation component includes neutrinos, which are nearly as numerous as CMB photons and are highly relativistic in the early universe.

The matter component includes dark matter, which appears to outweigh baryons by a factor $\sim 6 : 1$.

For $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$,

$$\begin{aligned}\Omega_{\gamma,0} &= 5.0 \times 10^{-5} \\ \Omega_{\nu,0} &= 3.4 \times 10^{-5} \\ \Omega_{r,0} &= \Omega_{\gamma,0} + \Omega_{\nu,0} = 8.4 \times 10^{-5} \\ \Omega_{\text{bary},0} &= 0.04 \\ \Omega_{\text{DM},0} &\approx 0.26 \\ \Omega_{m,0} &= \Omega_{\text{bary},0} + \Omega_{\text{DM},0} \approx 0.3.\end{aligned}$$

These are the numbers in the “benchmark model” described in §6.5 of the textbook. More recent observational data lead to slightly different best estimates of these values, but not by much.

With these numbers, $t_0 = 0.964/H_0$.

The value of $\Omega_{\gamma,0}$ is precisely known from the CMB temperature.

The cosmic neutrino background cannot be measured directly, but the value of $\Omega_{\nu,0}$ can be precisely calculated from theory given the standard model of particle physics. However, we have here treated neutrinos as *massless*, which is an excellent approximation in the early universe but not today.

The value of $\Omega_{\text{bary},0}$ is well determined (at the $\approx 10\%$ level) by measurements of the cosmic deuterium abundance and by measurements of anisotropy in the CMB.

The value of $\Omega_{\text{DM},0}$ is the most uncertain, but a variety of measurements imply it is probably known to 20%.

Based on these numbers, we conclude that radiation and matter had equal energy density at redshift

$$(1+z_{\text{eq}}) = \frac{\Omega_{m,0}}{\Omega_{r,0}} \approx 3570.$$

At redshifts much higher than z_{eq} , the universe was radiation dominated, with $a(t) \propto t^{1/2}$.

At z_{eq} there was a transition to a matter dominated universe, with $a(t) \propto t^{2/3}$.

At much lower redshift ($z \lesssim 2$), dark energy became important.

The age of the universe at z_{eq} is $t_{\text{eq}} = 4.7 \times 10^4$ yrs.