Astronomy 5682
Problem Set 2: Dark Matter in Galaxies
Due Thursday, January 31, in class

Question 1

The upper plot on the second-to-last page shows the rotation curve of the spiral galaxy NGC 3198, taken from a paper published in 1994. Crosses show measurements of the rotation velocity of neutral hydrogen gas as a function of radius, from radio observations. Open circles show rotation speed measurements from optical data, which better resolve the interior of the galaxy (because optical telescopes can usually achieve higher angular resolution than radio telescopes) but do not extend nearly as far (because the gas far from the center is not glowing at optical wavelengths). Filled circles show measurement of the dispersion of velocities at a given radius, but we will not make use of them in this problem. Instead, we will see what we can learn about dark matter in NGC 3198 from the rotation curve.

You can use Newtonian gravity and mechanics throughout this problem (though we will use GR light bending in Problem 2).

(a) Show that in a spherical mass distribution, a particle on a circular orbit at radius r moves with speed

$$v_c(r) = \sqrt{\frac{GM(r)}{r}}, \tag{1}$$

where $M(r)$ is the mass interior to radius r.

You may use without proof Newton's "iron sphere" theorem, which says that in a spherically symmetric mass distribution, the net gravitational force from mass exterior to radius r vanishes (i.e., there is no gravitational acceleration inside a hollow sphere).

(b) Observed galaxy disks typically have an exponential form, with surface density $\Sigma(r)$ given by

$$\Sigma(r) = \Sigma_0 e^{-r/r_s}, \tag{2}$$

where r_s is the exponential scale length and Σ_0 is the central surface density.

Show that an exponential disk has a total mass $M = 2\pi\Sigma_0 r_s^2$ and that the mass interior to radius r is

$$M(r) = \int_0^r \Sigma_0(r)\, 2\pi r\, dr = 2\pi\Sigma_0 r_s^2 \left[1 - e^{-y} - ye^{-y}\right], \tag{3}$$

where $y \equiv r/r_s$.

Hint: Integrate $\int xe^{-x}dx$ by parts with $dv = e^{-x}dx$.

If you can't solve this part in a reasonable amount of time, just assume equation (3) is true and move on.

(c) NGC 3198 has an exponential scale length $r_s = 2.7$ kpc and a total disk mass $M_{\rm disk} = 3 \times 10^{10} M_\odot$. In practice, we observe the exponential profile of the *light* and infer the underlying mass profile and total disk mass by assuming that the light is produced by a mix of stars similar to that found in the Milky Way.

Using equation (3) with this scale length and total mass, what is the amount of stellar mass interior to radius $r = 2$, 4, 10, 20, and 30 kpc?

(d) The observed rotation curve can be roughly described by the form $v_c(r) = 150\,\mathrm{km\,s^{-1}} \times (r/4\,\mathrm{kpc})$ for $r < 4\,\mathrm{kpc}$ and $v_c(r) = 150\,\mathrm{km\,s^{-1}}$ for $r = 4 - 30\,\mathrm{kpc}$. Draw these two lines onto the rotation curve graph to convince yourself that they make a decent approximation to the data.

Using this simple form of the rotation curve and equation (1), what is the implied total mass interior to $r = 2$, 4, 10, 20, and 30 kpc?

Make a plot with your values from part (c) and part (d).

You will find it convenient to use G in the units

$$G = 4.36 \times 10^{-6} \left(\mathrm{km\,s^{-1}}\right)^2 \mathrm{kpc}\, M_\odot^{-1}. \tag{4}$$

(e) Assuming that Newtonian gravity (which gives predictions indistinguishable from GR on these scales) is correct, what can you conclude about the presence and spatial distribution of dark matter in NGC 3198? Suppose that you allowed stars in NGC 3198 to be heavier than they are in the Milky Way, with more mass for the same amount of light, but still assumed they are distributed in an exponential disk with $r_s = 2.7\,\mathrm{kpc}$. Could this remove the need for dark matter to explain the observed rotation curve?

Question 2

General Relativity predicts that a light ray passing by a mass M with an impact parameter b will be bent by an angle

$$\alpha = \frac{4GM}{c^2 b} \text{ radians.} \tag{5}$$

The lower figure on the second-to-last page, taken from a 2006 paper by Adam Bolton and collaborators, shows an image of an "Einstein ring." The dark blob in the middle is an elliptical galaxy, with a redshift of 0.32. The ring around it is the gravitationally lensed image of a background galaxy, with redshift 0.52.

This object was identified as interesting because its spectrum measured by the Sloan Digital Sky Survey (SDSS) appeared to be the superposition of two spectra at different redshifts. The image comes from the ACS camera on Hubble Space Telescope. If you do a Google Image search for "SLACS Einstein Ring" you will find many more examples of such rings, and color images that highlight the difference in color between the two galaxies.

(a) Suppose we observe a galaxy at a distance D that has another galaxy exactly behind it at a distance $2D$ (i.e., twice as far from us as the original galaxy). What is the expected appearance of this galaxy pair on the sky?

Draw a diagram to illustrate your answer, indicating the angle α and the impact parameter b.

(b) The ACS image is 8 arcsec by 8 arcsec. With a ruler, estimate the value of α in arcsec from the image.

For this part and the remainder of the problem, assume that the background galaxy is indeed at twice the distance of the foreground galaxy. This is not exactly true in this case, but it's reasonably close.

(c) For reasonable cosmological parameters, the distance (specifically the "angular diameter distance") to a galaxy at redshift 0.32 is 960 Mpc. Based on this distance, what is the value of the impact parameter b, in kpc?

Remember that there are 206,265 arcsec/radian.

Extra Credit (2 points): Compare 960 Mpc to the "naive" value you get from applying Hubble's law with $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Why don't the values agree exactly?

(d) Using equation (5) and your results in (b) and (c), what is the value of

$$v_c(r = b) = \sqrt{\frac{GM}{b}} \quad ?$$

What is the mass M interior to $r = b$?

(e) Stars in an ellliptical galaxy are not on circular orbits, so one can't measure a rotation curve. However, one can measure the velocity dispersion — the rms spread of stellar velocities along the line of sight — from the width of absorption lines in the spectrum. For reasonable assumptions about the stellar orbits, the relation between this dispersion σ and mass turns out to be the same as the one for circular velocity and mass:

$$\sigma^2(r) = \sqrt{\frac{GM(r)}{r}} \quad . \tag{6}$$

The velocity dispersion of this galaxy, measured from the SDSS spectrum, is $\sigma = 310\,\mathrm{km\,s^{-1}}$ (with an uncertainty of $15\,\mathrm{km\,s^{-1}}$). The radius r_v of the visible region of the galaxy on the ACS image is about half the radius of the Einstein ring. With this value of σ, what is the implied mass interior to r_v? How does this compare to the mass interior to $r = b$ from part (d)?

(f) Based on your results from (d) and (e) and the visual appearance of the image, what can you say about dark matter in this galaxy?

Extra Credit (5 points): Why is the survey that yielded this object called SLACS? If you look at the online color images, you will see that the light from the background "ring" galaxy is bluer than that from the foreground, lens galaxy, even though the background galaxy is at higher redshift. Why?