Astronomy 5682
Problem Set 3: Measuring Curvature
Due Thursday, Feb 7, in class

Question 1: Circles on a Sphere (30 points)

(This is Problem 3.3 from the textbook.)

Suppose that you are a two-dimensional being living on the surface of a sphere of radius R. Show that if you draw a circle of radius r, the circle's circumference will be

$$C = 2\pi R \sin(r/R). \tag{1}$$

Idealize the Earth as a perfect sphere of radius $R = 6731$ km. If you could measure distances with an error of ± 1 meter, how large a circle would you have to draw on the Earth's surface to convince yourself that the Earth is spherical rather than flat?

Hint: After writing down the necessary condition, you may want to use the Taylor expansion of $\sin(r/R)$ to get an equation you can solve analytically. Alternatively, you can solve the equation numerically by trial and error.

Question 2: Angular Sizes in an Expanding Universe (40 points)

Recall that:

- The FRW metric is

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_k^2(r) d\Omega^2\right],$$

where

$$\begin{aligned} S_k(r) &= R_0 \sin(r/R_0), \qquad k = +1, \\ &= r, \qquad\qquad\qquad k = 0, \\ &= R_0 \sinh(r/R_0), \qquad k = -1. \end{aligned} \tag{2}$$

- The comoving distance to an object that emitted light at time t_e is

$$r = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} \tag{3}$$

- The cosmological redshift is $1 + z = \frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e}$, and by definition $a_0 = a(t_0) = 1$.

(a) Consider a universe that has $a(t) = (t/t_0)^{2/3}$, flat space geometry ($k = 0$), and $t_0 = 14$ Gyr (14 billion years). What is the comoving distance r to a galaxy at redshift $z = 3$? Express your answer in light years.

(b) Suppose this galaxy is oriented perpendicular to the line of sight and subtends an angle $d\Omega = 3.6$ arc-seconds (10^{-3} degrees). What is the galaxy's physical size, in light years?

(c) Consider instead a universe that has $a(t) = t/t_0$, negative space curvature ($k = -1$), a curvature radius $R_0 = ct_0$, and $t_0 = 14$ Gyr. What is the comoving distance r to a galaxy at $z = 3$?

(d) Suppose this galaxy is oriented perpendicular to the line of sight and subtends an angle $d\Omega = 3.6$ arc-seconds (10^{-3} degrees). What is the galaxy's physical size, in light years?

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(e) Suppose you observed an object (maybe a galaxy, maybe something else) of known physical size and measured its redshift, $z = 3$. How could you use the observed angular size of the object to decide which of the two models (the one in a/b or the one in c/d) better describes the real universe? In which kind of universe would the object have the larger angular size?

Question 3: Constraining Cosmic Curvature (30 points)

Galaxies do not come with their true physical sizes conveniently labeled, so the test of 2(e) cannot be carried out in practice. However, there *is* a characteristic scale imprinted on the *clustering* of galaxies in space known as the baryon acoustic oscillation (BAO) feature, which I'll explain further in class. The size of the BAO feature can be calculated from known physics, and for this problem we'll take it to be exactly 500 million light years in *comoving* units — i.e., the physical size at redshift z is $500(1+z)^{-1}$ million light years.

Suppose that you measure the angular scale of the BAO feature from the clustering of galaxies at redshift $z = 0.5$ and find a result that is consistent with the expectation for a flat ($k = 0$) universe, with a measurement uncertainty of 2%. (The Baryon Oscillation Spectroscopic Survey recently did something very close to this.)

(a) Show that in a universe with $a(t) = (t/t_0)^{2/3}$ the comoving distance to $z = 0.5$ is $r = 0.379ct_0$.

(b) Suppose that the value $t_0 = 14\,\text{Gyr}$ is known exactly, and that $k = 0$. What is the expected angular size corresponding to the 500 million-light-year BAO scale? Give your answer in degrees.

(c) Your measurement is consistent with this prediction, but with 2% uncertainty. What *lower limit* can you set on the value of the curvature radius R_0, assuming that this is the only source of uncertainty in the problem?

Hint: See Question 1.

(d) You decide to carry out a larger survey to improve the precision of your measurement, and you have two choices: a survey that will yield a 1% measurement at $z = 0.5$ or a survey that will yield a 2% measurement at $z = 1$. Which of these will give you a better constraint on curvature (i.e., a stronger lower limit on R_0 assuming the measurement is still compatible with flatness)?