Astronomy 5682
Problem Set 4: Solutions to the Friedmann Equation
Due Thursday, Feb 14, in class

Question 1: A Matter-Dominated Universe (40 points)

As discussed in class and in Chapter 4 of the textbook, the Friedmann Equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2}\frac{1}{a^2(t)}.$$

(a) Explain why a universe with present energy density

$$\frac{\epsilon_0}{c^2} = \frac{3H_0^2}{8\pi G}$$

must have $k = 0$.

The quantity $\rho_{\rm crit} = 3H_0^2/(8\pi G)$ is referred to as the critical density.

(b) Argue that if the dominant form of energy in the universe is non-relativistic matter (i.e., atoms or particles whose kinetic energy is negligible compared to their rest mass energy) then

$$\epsilon(t) = \epsilon_0 \left[a(t)\right]^{-3}.$$

(c) Show that in a critical density, matter-dominated universe, the evolution of the expansion factor is given by

$$a(t) = (t/t_0)^{2/3}.$$

(d) Show that the age of a critical density, matter-dominated universe is

$$t_0 = \frac{2}{3} \times \frac{1}{H_0}.$$

[*Hint:* Recall that $H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt}$, and $t_0 = \int_{a=0}^{a=1} dt$.]

(e) Show that the comoving distance to an object at redshift z in a critical density, matter dominated universe is

$$r = \frac{2c}{H_0}\left[1 - a_e^{1/2}\right],$$

where $a_e = (1+z)^{-1}$ is the expansion factor at the time the object emitted its light.

Question 2: An Empty Universe (20 points)

Consider an empty universe, with energy density $\epsilon = 0$.

(a) What is the curvature index k?

(b) What is the solution for $a(t)$?

(c) What is the relation between t_0 and H_0?

(d) What is the comoving distance to an object at $z = 1$?

Question 3: A Lambda-Dominated Universe (40 points)

Consider a spatially flat ($k = 0$) universe in which the only energy component is a cosmological constant, with energy density $\epsilon_\Lambda = \epsilon_{c,0}$ that *does not change* as the universe expands.

(a) Show that the solution to the Friedmann equation in this case is

$$a(t) = a_0 e^{H_0(t-t_0)}$$

(b) What is the age of the universe (in terms of H_0 and t_0) at redshift $z = 1$?

(c) What is the comoving distance to an object at $z = 1$?

(d) What is the physical separation of two objects separated by $d\Omega = 0.01$ radians at $z = 1$?

(e) Consider a photon emitted today (at t_0). What comoving distance r has it traveled to by time $t_f > t_0$?

(f) Suppose that this model is a good description of our universe (it can't be a perfect description, because it has no matter). If a supernova goes off in our galaxy today, will an observer in a galaxy that is presently 6000 Mpc away from us ever be able to see it?

(Assume that, as in our universe, $c/H_0 \approx 4300$ Mpc.)

(g) Does this universe have a "big bang"?