Astronomy 5682
Problem Set 7: The Geometry of Space, $\Omega_{m,0}$, and $\Omega_{\Lambda,0}$
Due Thursday, April 11

Overview

The curvature of space depends on Ω_0, the sum of the matter density $\Omega_{m,0}$ and the vacuum energy density $\Omega_{\Lambda,0}$. (In this problem set, we will ignore the contribution of radiation and assume that dark energy, if it exists, is constant in time.) If the Cosmological Principle holds for the entire universe (not just the part that we can see), then a universe with flat or negatively curved space is infinite, while a universe with positively curved space is finite. So if we can determine the geometry of space, we can get some idea of whether the universe is finite or infinite in extent.

We can measure the curvature of space if we can determine the angular size of a distant (high redshift) structure of known physical size. In this problem set, we will use the angular sizes of "hot spots" in a map of the cosmic microwave background (hereafter abbreviated CMB) to try to decide whether we live in a flat universe with $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$ or a negatively curved universe with $\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0$. Finally, we will combine the results with the supernova results to estimate $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$.

Part I: A Scale in the Cosmic Microwave Background

There is, in fact, a structure in the distant universe whose physical scale we can predict theoretically: the scale of the strongest fluctuations in the CMB. For reasons that we will briefly discuss in class, this scale is the distance that a "sound wave" (which in this case means any pressure-driven disturbance) could travel before the universe becomes neutral at redshift $z_{\rm rec} = (3000\ {\rm K})/(2.7\ {\rm K}) \approx 1100$. Structures of this size appear as "hot spots" in sensitive maps of temperature fluctuations in the CMB, so their angular size can be measured observationally. The subscript "rec" stands for recombination, the formation of neutral hydrogen, and after $z_{\rm rec}$ the universe is transparent to CMB photons, so no new structure can be imprinted on the CMB.

You will need to remember our equations for the Hubble parameter

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{(1-\Omega_0)}{a^2}.$$

and the comoving distance

$$r = \frac{c}{H_0}\int_0^z \frac{dz'}{\left[\Omega_{r,0}(1+z')^4 + \Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0} + (1-\Omega_0)(1+z')^2\right]^{1/2}}.$$

At the redshifts of interest in this problem set, we should not completely ignore the contribution of the $\Omega_{r,0}$ term, but we will do so nonetheless.

(a) First consider the case $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$. Show that age of the universe at redshift $z_{\rm rec}$ is

$$t_{\rm rec} = \frac{2}{3H_0}(1+z_{\rm rec})^{-3/2}.$$

For $H_0 = 70\ {\rm km\,s^{-1}\,Mpc^{-1}} \approx (14\ \text{billion years})^{-1}$, what is $t_{\rm rec}$?

(Note that this will be somewhat different from the true value because we have assumed $\Omega_{m,0} = 1$ and $\Omega_{r,0} = 0$.)

(b) Before z_{rec}, photons are constantly scattering off of electrons, and one can show that the "speed of sound", denoted c_s, is

$$c_s = \frac{c}{\sqrt{3}} \approx \frac{c}{1.732},$$

where c is the speed of light.

How far could a sound wave travel in time t_{rec}? Make life easy by expressing your answer in light years.

(c) The characteristic size of "hot regions" and "cold regions" at t_{rec} is roughly $2c_s t_{\text{rec}}$. Argue that the characteristic angular size of hot spots and cold spots on the CMB should therefore be

$$\theta_c = \frac{2c_s t_{\text{rec}}}{S_k(r)}(1 + z_{\text{rec}}),$$

where r is the comoving distance.

Hint: Refer to the FRW metric.

(d) Show that for $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$, the comoving distance to z_{rec} is, to an accuracy of a few percent,

$$r \approx \frac{2c}{H_0}.$$

What is the predicted value of θ_c for $\Omega_{m,0} = 1$ and $\Omega_{\Lambda,0} = 0$, in degrees?

(e) Suppose instead that $\Omega_{m,0} = 0.3$. Argue that, in this case, the age of the universe at z_{rec} is approximately

$$t_{\text{rec}} = (\Omega_{m,0})^{-1/2} \frac{2}{3H_0}(1 + z_{\text{rec}})^{-3/2}.$$

(f) Calculating r for a universe with $\Omega_{m,0} = 0.3$ requires doing the integral numerically. The result is $r = 3.2c/H_0$ for a flat universe with $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$.

What is the predicted value of θ_c for this case?

(g) For an open (negatively curved) universe with $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0$, numerical evaluation of the integral gives $r = 2.8c/H_0$.

What is the predicted value of θ_c for this case?

You will need to remember the formula for the curvature radius,

$$R_0 = \frac{c}{H_0}(1 - \Omega_0)^{-1/2}.$$

Part II: Measurement

The upper figure on the next page is a map of the CMB from the BOOMERANG experiment (you can find more about this experiment by Googling "Boomerang CMB"). While the "spots" on the map are irregular and varied, they do have a characteristic size.

(h) Choose eight strong spots from the central area of the map (enclosed by the curved line) and measure their size in mm. Specifically, you should measure the size of the yellow region, going across the shortest dimension if the spot is elongated. Avoid the three small, circled spots, which are radio galaxies rather than CMB spots.

What is the average size of the spots in mm?

(i) Using the scale marked in degrees along the *vertical* axis (labeled DEC; don't use the horizontal axis labeled RA), convert your average size to degrees. (Extra Credit [2 points]: Why shouldn't you use the RA axis?)

What is the typical size of CMB hot spots measured by BOOMERANG, in degrees? Is this number closer to your answers for a flat universe, from (d) and (f), or for a negatively curved universe, from (g)?

(j) The lower figure on the next page is the CMB power spectrum measured by BOOMERANG. From this plot, estimate the value of l at the *first* (highest) peak in the power spectrum and convert it to an angular scale using the formula $\theta \approx 200/l$ degrees. What is the value of $\theta_{\rm peak}$? What is the *uncertainty* in your measurement (i.e., assuming that the data in the plot are correct, how precisely can you measure $\theta_{\rm peak}$ from it)? Is your result from this measurement consistent with your "spot" measurement from (i)?

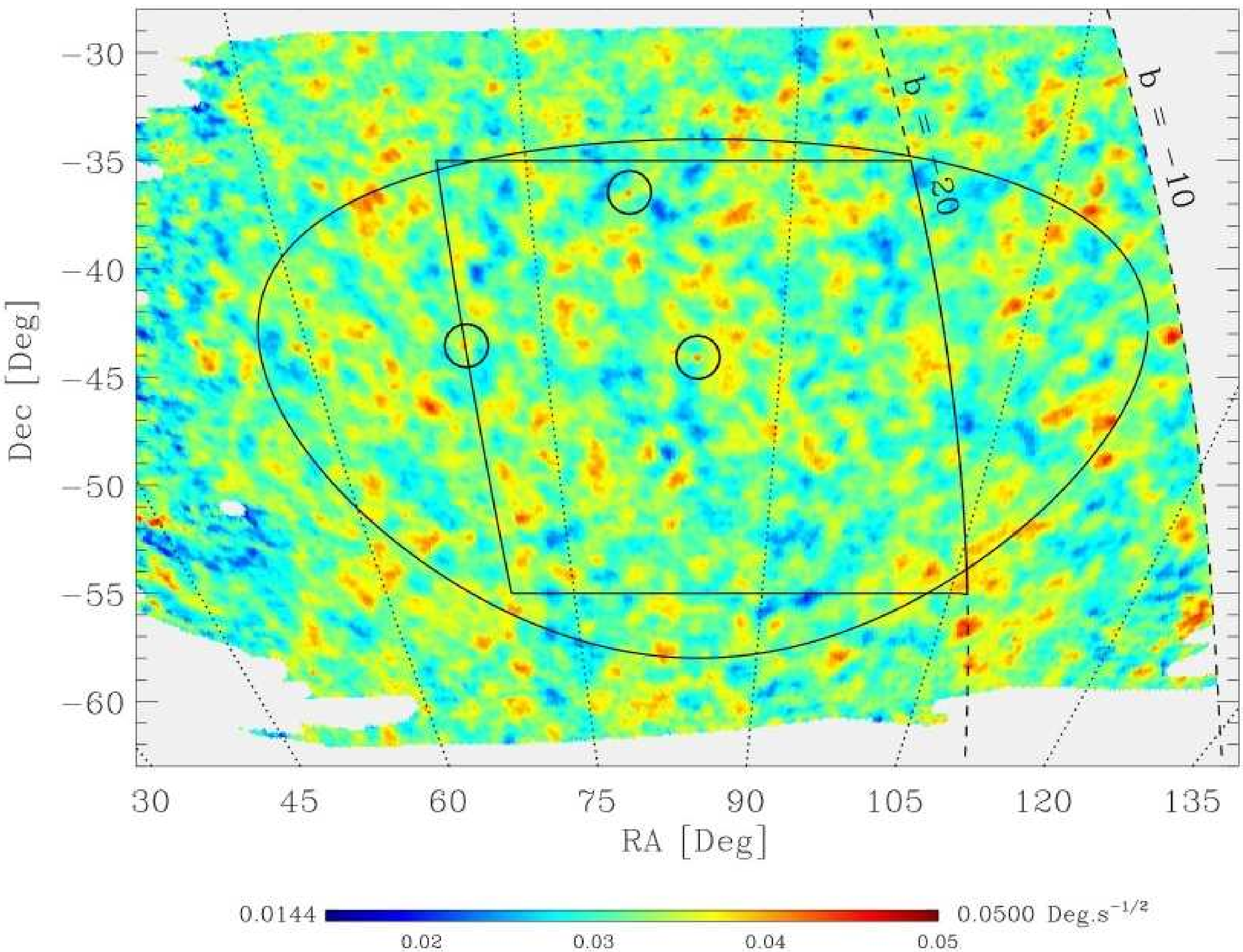
Part III: Pinning Down $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$

A more careful analysis of this map, and similar maps from other experiments, shows that $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$, with an uncertainty of about 0.03. The sum of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ determines the curvature of space because the energy density of matter and vacuum energy act together to produce curvature.

The evolution of the expansion rate, however, is governed (approximately) by the *difference* between matter density and vacuum energy density, $\Omega_{m,0} - \Omega_{\Lambda,0}$, because matter acts to decelerate the expansion of the universe and vacuum energy acts to accelerate it. The supernova results that you analyzed in Problem Set 5 can be well fit by a universe with $\Omega_{m,0} - \Omega_{\Lambda,0} = -0.4$.

(k) Draw a graph in which the horizontal axis is $\Omega_{m,0}$ and the vertical axis is $\Omega_{\Lambda,0}$, with the range of each axis running from zero to one. Label the axes. On this graph, draw the line $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$. Also draw the line $\Omega_{m,0} - \Omega_{\Lambda,0} = -0.4$.

If the CMB results and the supernova results are both right, what can you conclude about the values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$?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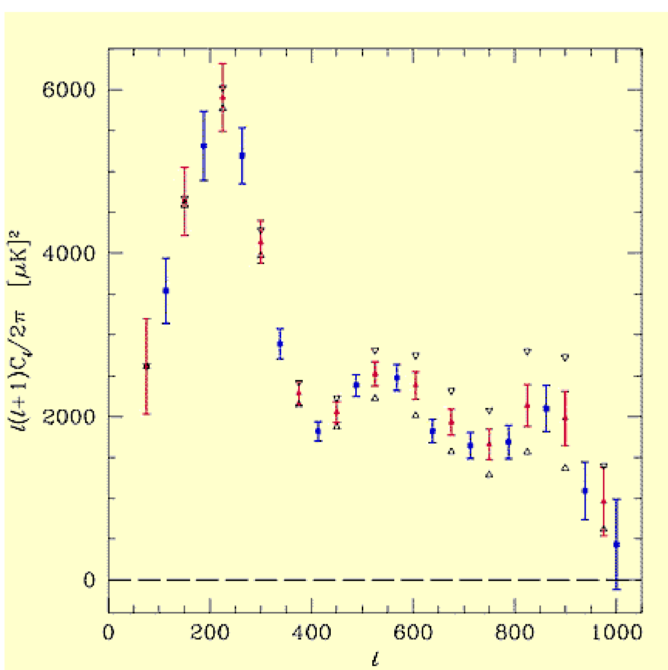
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