Problem Set 8: Inflation
Due Thursday, April 18

For Questions 1 and 2, you can work at an order-of-magnitude level, so you don't need to keep track of factors of 2, π, etc. However, you should keep track of these factors in Question 3. If you haven't already read Chapter 11, you should do so before doing the problem set.

Question 1: The Horizon Problem (40 points)

For a start, assume a standard FRW universe, which at early times is radiation dominated. As you showed in Problem Set 6, the relation between the age of the universe and temperature during this early phase is

$$t(T) \approx 1 \left(\frac{kT}{\text{MeV}}\right)^{-2} \text{ s.} \tag{1}$$

As explained in class, a photon emitted at time $t = 0$ can travel a physical distance $2ct$ by time t in a radiation-dominated universe. (See also Problem Set 4, Question 1e, though that was for a matter dominated universe.) For this problem, we will just approximate this as ct and refer to it as the horizon distance.

(a) The temperature T_{GUT}, where "grand unification" of the strong and electroweak forces is thought to occur, is believed (on the basis of particle physics arguments) to be $kT_{\text{GUT}} \approx 10^{18}$ MeV $= 10^{24}$ eV.

What is the horizon distance $d_{\text{hor}}(T_{\text{GUT}})$ at the time when $T = T_{\text{GUT}}$? Express your answer in light-seconds.

(b) What is the present-day physical distance d_{LSS}, to the last scattering surface, the source of the cosmic microwave background, in light-seconds? For purposes of this problem, you can approximate this distance as $3ct_0$ (you don't have to justify this in your answer, though you should understand where it comes from), and you can round your answer to the nearest order-of-magnitude.

(c) What was the *physical* (not comoving) size of d_{LSS} at the time when $T = T_{\text{GUT}}$? Express your answer both in light-seconds and in cm.

Hint: Use the fact that the present-day temperature of the CMB is $kT_{\text{CMB}} \approx 10^{-4}$ eV to decide how much the universe has expanded since t_{GUT}.

(d) If all went well, the value you found in (a) is about 26 orders of magnitude smaller than the answer in (c). Explain the connection between this 26 order-of-magnitude discrepancy and the "horizon problem" of the standard big bang model.

(e) Suppose that when $T \approx T_{\text{GUT}}$ the universe enters a period of inflation. During inflation the universe expands as $a(t) \propto e^{Ht}$, and the temperature remains at $T = T_{\text{GUT}}$ instead of dropping as a^{-1}. (More accurately, the radiation temperature drops during inflation, but it is restored to $T \approx T_{\text{GUT}}$ at the end of inflation, when the inflaton field decays and converts its energy into photons and other relativistic particles.)

During inflation, the universe expands by a total factor e^N. What is the minimum value of N required to solve the horizon problem, i.e., to ensure that the entire region out to the last scattering surface was in one causally connected patch before inflation began?

From now on, we will refer to this minimum number of e-folds as N_{min}.

Question 2: The Flatness Problem (30 points)

Suppose that before inflation begins the gravitational and curvature terms in the Friedmann equation are of similar order, and therefore

$$H^2 \sim \frac{8\pi G}{3c^2}\epsilon \sim \frac{c^2}{a^2 R_0^2}. \tag{2}$$

(a) What is the approximate value of the curvature radius aR_0 at time $t_{\rm GUT}$, just before inflation begins?

(b) If the number of e-folds is the minimum number $N_{\rm min}$ required to solve the horizon problem, what is the curvature radius today?

(c) Is it possible for inflation to solve the horizon problem and still produce a universe with Ω measurably different from one today? Is it likely for this to happen? (Here Ω represents the contribution of all energy components — matter, radiation, dark energy.)

Question 3: The Initial Conditions for Inflation (30 points)

Assume that during inflation, the energy density is dominated by the potential energy $V(\phi)$ of a scalar field ϕ. Instead of the case considered in the book (Figure 11.3), where the field ϕ starts near zero and rolls to the minimum of $V(\phi)$ at some non-zero ϕ, we'll assume a potential

$$V(\phi) = (\hbar c)^{-3}\lambda\phi^4, \tag{3}$$

with a minimum at $\phi = 0$. λ is dimensionless, and the factor $(\hbar c)^{-3}$ converts the energy^4 units of ϕ^4 to an energy per unit volume.

We assume that ϕ starts at some initial value ϕ_i, and inflation occurs while ϕ rolls to its minimum. The goal of this problem is to deduce the value of ϕ_i required to get the minimum number of e-folds deduced in Question 1.

You'll need to use the Friedmann equation during the inflationary epoch,

$$H^2 = \frac{8\pi G}{3c^2}V(\phi), \tag{4}$$

and the slow-roll equation discussed in class and in the textbook,

$$3H\dot{\phi} = -\hbar c^3 V'(\phi), \tag{5}$$

where $V'(\phi) = dV/d\phi$.

(a) Define t_i to be the time at which inflation starts and ϕ has the value ϕ_i, and $t_{\rm end}$ to be the time at which $\phi = 0$ and inflation ends. Argue that the number of e-folds of inflation during this evolution is

$$N = \int_{t_i}^{t_{\rm end}} H(t)dt = \int_{\phi_i}^{0} H(\phi)dt = \frac{8\pi G}{c^2}(\hbar c^3)^{-1}\int_0^{\phi_i}\frac{V(\phi)}{V'(\phi)}d\phi. \tag{6}$$

You should give verbal arguments that justify the first two equalities and a mathematical derivation that justifies the third. (Hint: use the relation between dt, $\dot{\phi}$, and $d\phi$.)

(b) Now assume the specific form of $V(\phi)$ given in equation (3). What value of ϕ_i is required to get the minimum number of e-folds that you derived in 1(e).

(c) Inflation models with this generic form of $V(\phi)$ are usually referred to as "large field" inflation models because the initial value of ϕ must be larger (by a factor of several) than a physically interesting energy scale. What is this energy scale called, and why is it interesting?