II. General Relativity

Reading: Chapter 3 (Sections 3.1 and 3.2)

Special Relativity

Postulates of theory:

1. There is no state of "absolute rest."
2. The speed of light in vacuum is constant, independent of state of motion of emitter.

(Point 2 could really be subsumed into point 1.)

Implies: Simultaneity of events and spatial separation of events depend on state of motion of observer.

Relation between coordinate systems x, y, z, t and x', y', z', t' of uniformly moving observers is described by Lorentz transformations.

For a reference frame moving at constant velocity v in $+x$ direction, the *Galilean* transformation is

$$\begin{aligned} t' &= t \\ x' &= (x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

The *Lorentz* transformation is

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z, \end{aligned}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the *Lorentz factor.*

Derived by considering a spherical light wave emitted at $t = t' = 0$, for which

$x^2 + y^2 + z^2 = c^2t^2$ must imply $x'^2 + y'^2 + z'^2 = c^2t'^2$

if c is observer independent.

Observers in relative motion disagree on the spatial separation $\Delta l = \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]^{1/2}$ and the time separation Δt between the same pair of events.

But they agree on the "spacetime interval" $(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ between events.

Analogy: Stationary observers with rotated reference frames disagree on Δx, Δy but agree on $\Delta l = \left[(\Delta x)^2 + (\Delta y)^2\right]^{1/2}$.

The Equivalence Principle

"Special" relativity restricted to uniformly moving observers. Can it be generalized?

Newtonian gravity: $\mathbf{a} = \mathbf{F}/m$, $\mathbf{F} = -GMm\hat{\mathbf{r}}/r^2$.
Why is this unsatisfactory?

Implicitly assumes infinite speed of signal propagation.

Coincidental equality of inertial and gravitational mass.

Einstein, 1907: "The happiest thought of my life." If I fall off my roof, I feel no gravity.

Equivalence between uniform gravitational field and uniform acceleration of frame. True in mechanics. *Assume* exact equivalence, i.e., for electrodynamics as well.

Equivalence principle implies gravitational and inertial masses *must* be equal.

Allows extension of relativity to accelerating frames.

Implies that extension of relativity *must* involve gravity.

Third frame trick $\longrightarrow$ gravitational mass of electromagnetic energy, gravitational redshift and time dilation, bending of light (incorrect answer because ignores curvature of space)

Restatement of equivalence principle: In the coordinate system of a freely falling observer, special relativity always holds locally (to first order in separation). No gravity.

Over larger scales (second order in separation), gravity doesn't vanish in a freely falling frame — tidal effects. E.g., freely falling objects in an inhomogeneous gravitational field may accelerate towards or away from each other.

Summary of General Relativity

With aid of equivalence principle, can change relativity postulate from "There is no absolute rest frame" to "There is no absolute set of inertial frames."

More informally, "There is no absolute acceleration."

Uniform acceleration can be treated as uniform gravitational field; the two are *indistinguishable.*

A *geodesic path* is a path of shortest distance, e.g.,

In flat space, a straight line.

On a sphere, a great circle.

In relativity, a geodesic path is a path of shortest *spacetime interval.*

In flat spacetime, a straight line at constant velocity.

Freely falling particles move along these geodesics in flat spacetime.

GR description of gravity:

All freely falling particles follow geodesic paths in curved spacetime.

Distribution of matter (more generally, stress-energy) determines spacetime curvature.

Misner, Thorne, and Wheeler's catchy summary of GR:
Spacetime tells matter how to move. (Along geodesic paths.)
Matter tells spacetime how to curve. (Field equation.)

Compare to equivalent description of Newtonian gravity:
Gravitational force tells matter how to accelerate. ($F = ma$.)
Matter tells gravity how to exert force. ($F = GMm/r^2$.)

The "equivalence of inertial and gravitational mass" in Newton's description is not a coincidence but a *necessary consequence* of the assumption that all freely falling particles follow geodesic paths.

Space curvature and the spatial metric

On a flat, two-dimensional surface, the angles of a triangle satisfy

$$\alpha + \beta + \gamma = \pi.$$

The spatial separation dl between two nearby points is

$$dl^2 = dx^2 + dy^2,$$

or, in polar coordinates,

$$dl^2 = dr^2 + r^2 d\theta^2.$$

The total length l of a path can be found by integrating dl along the path.

On the surface of a sphere, the angles of a triangle add to

$$\alpha + \beta + \gamma = \pi + A/R^2,$$

where A is the area enclosed by the triangle and R is the radius of the sphere.
If r is the spatial distance along a great circle from the origin (e.g., the North Pole) and θ the azimuthal angle (e.g., the longitude), then the spatial separation of nearby points is

$$dl^2 = dr^2 + R^2 \sin^2(r/R) d\theta^2.$$

Analogously, on a negatively curved (saddle-like) surface of constant curvature, the angles of a triangle add to

$$\alpha + \beta + \gamma = \pi - A/R^2,$$

and the spatial separation is

$$dl^2 = dr^2 + R^2 \sinh^2(r/R) d\theta^2.$$

Formulas that relate coordinate separations to length separations are called *metrics*.

These results for two-dimensional surfaces can be naturally generalized three-dimensional spaces.

The metrics for flat, positively curved, and negatively curved spaces of constant curvature radius R, in "spherical" coordinates, are, respectively,

$$dl^2 = dr^2 + r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right]$$
$$dl^2 = dr^2 + R^2\sin^2(r/R)\left[d\theta^2 + \sin^2\theta d\phi^2\right]$$
$$dl^2 = dr^2 + R^2\sinh^2(r/R)\left[d\theta^2 + \sin^2\theta d\phi^2\right].$$

Positive curvature $\Longrightarrow$geodesics "accelerate" (in 2nd derivative sense) towards each other. Initially "parallel" geodesics converge.

Example: great circles on a sphere.

Zero curvature $\Longrightarrow$no geodesic "acceleration." Initially parallel geodesics stay parallel. Euclidean geometry.

Example: straight lines on a plane.

Negative curvature $\Longrightarrow$geodesics "accelerate" away from each other. Initially parallel geodesics diverge.

Example: geodesics on a saddle.

Spacetime metric

In GR, as in differential geometry, a fundamental role is played by the *metric tensor* $g_{\mu\nu}$.

The indices μ, ν run from 0 to 3, representing the time coordinate and three spatial coordinates.

The metric tensor is symmetric, so it has 10 independent components rather than 16.

Spacetime interval between two events separated by small dx^μ is

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu}dx^\mu dx^\nu.$$

In special relativity,

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2,$$

so $g_{\mu\nu} = \mathrm{diag}(-c^2, 1, 1, 1)$.

This is called the "Minkowski metric."

Note that even in flat spacetime, we would change the metric if we went to, e.g., polar coordinates, though they would still describe the same underlying geometry.

ds^2 is independent of coordinate system, hence state of motion of observer.

$ds^2 < 0: \ |ds| =$ proper time measured by an observer passing through events

$ds^2 > 0: \ |ds| =$ distance measured by an observer who sees both events as simultaneous

Can integrate $s = \int ds$ to get interval along a path between widely separated events.

Observers with different coordinate systems (e.g., moving relative to each other in arbitrary ways) disagree on values of dx^μ, $g_{\mu\nu}$.
All agree on value of ds^2.
Note that $g_{\mu\nu}$ is, in general, a function of spacetime position.

General Relativity Equations

Mathematically, GR is defined by "two" equations.
The first is the equation for geodsic paths, which gives the equation of motion for freely falling particles (or photons) in a specified coordinate system.
In practice, this equation represents four 2nd-order differential equations that determine $x^\alpha(s)$, given the initial position and 4-velocity.
The geodesic equation is the relativistic analog of the Newtonian equation $\mathbf{g} = -\vec{\nabla}\Phi$.
The metric $g_{\mu\nu}$ is the relativistic generalization of the gravitational potential.

The second is the *Einstein Field Equation*, which relates the curvature of spacetime to the distribution of matter and energy.
This is the analog of the Newtonian equation for the gravitational potential, which is often written in differential form as the *Poisson Equation*: $\nabla^2\Phi = 4\pi G\rho$.
The Field Equation is usually written

$$G_{\mu\nu} = 8\pi G_{\rm Newton} T_{\mu\nu}.$$

$G_{\mu\nu}$ is the *Einstein tensor*, built from $g_{\mu\nu}$ and its derivatives up to second order. (Like $\nabla^2\Phi$.)
$T_{\mu\nu}$ is the *stress energy tensor*, the relativistic generalization of density.
For an ideal fluid at rest, $T_{\mu\nu} = {\rm diag}(\rho, p/c^2, p/c^2, p/c^2)$.
The constant $8\pi G_{\rm Newton}$, where $G_{\rm Newton}$ is Newton's gravitational constant, is determined by demanding correspondence to Newtonian gravity in the appropriate limit.

Solutions of the field equation

Note that $G_{\mu\nu} = 8\pi G_{\rm Newton} T_{\mu\nu}$ is a set of ten, second-order differential equations for the ten components of $g_{\mu\nu}$.
Second-order $\Longrightarrow$

boundary conditions matter

spacetime can be curved even where $T_{\mu\nu} = 0$

propagating wave solutions exist

Nonlinear $\Longrightarrow$hard to solve.

Some exact solutions, e.g.

$\mathbf{T} = 0$ everywhere $\longrightarrow$ flat spacetime, "Minkowski space"
Spherically symmetric, flat at ∞, point mass at $r = 0 \longrightarrow$ Schwarzschild solution
Generalization to include angular momentum $\longrightarrow$ Kerr solution
Homogeneous cosmologies, which we will study

In other cases, approximate, by considering small departures from an exact solution (perturbation theory).

Recall that the (Newtonian) gravitational potential Φ has units of velocity^2.

The "weak field" limit of GR corresponds to $\Phi \ll c^2$. Spacetime curvature is weak; photons travel on nearly straight paths.

The combination of the weak field limit and $v \ll c$ leads to the Newtonian limit, in which GR approaches Newtonian gravity.

In this limit, only the 00 (time-time) component of the geodesic equation is non-trivial. It yields

$$\mathbf{g} = \frac{1}{2}\vec{\nabla} g_{00}$$

for the gravitational acceleration $\mathbf{g}$ of a freely falling particle.

Thus, g_{00} can be identified with -2Φ, where Φ is the Newtonian gravitational potential.

The 00 component of the Einstein field equation leads to

$$\nabla^2 g_{00} = 8\pi G_{\rm Newton}(\rho + 3p/c^2).$$

For a non-relativistic fluid, $p \ll \rho c^2$, and we get the equation of motion of a particle moving under the influence of a gravitational potential Φ that obeys Poisson's equation.

Tests of GR

- yields Newtonian gravity in appropriate limit
- precision tests of equivalence principle
- precession of Mercury – the key from Einstein's point of view
- bending of light – historically important
- gravitational redshift
- higher-order solar system tests $\Longrightarrow$measured values of "post-Newtonian parameters" agree with GR predictions
- binary pulsars:

  gravity wave dissipation rate – very strong test
  precession of orbit in an external system
  gravitational time delay, effects up to $\sim (v/c)^3$

Other low precision tests: structure of dense stars, gravitational lensing

Despite these impressive tests, application to cosmology requires gigantic extrapolation in length and time scale.

Can't rest comfortably on empirical basis of small-scale tests.

Cosmological models based on GR are impressively successful, but they require two strange ingredients: dark matter and dark energy.

Existence of these ingredients could be an indication that GR is breaking down in some way on cosmological scales, though we will generally take the view that it is not.