V. Cosmic Expansion History

Reading: Chapters 5, 6, and 7

Solutions of the Friedmann Equation

We now understand the origin of the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2}\frac{1}{a^2(t)},$$

and the dependence of energy density $\epsilon(t)$ on the expansion factor $a(t)$ for various forms of matter and energy.

Now we want to know what the *solutions* of the Friedmann equation are for interesting cases.

For single-component universes, the following cases can be derived by integrating the Friedmann equation or verified by substitution. You are doing three of them in Problem Set 4.

Matter-dominated, critical density ($k = 0$)

$$\epsilon = \epsilon_0(a/a_0)^{-3}, \qquad a(t) = (t/t_0)^{2/3}, \qquad H = 2/3t.$$

Radiation-dominated, critical density ($k = 0$)

$$\epsilon = \epsilon_0(a/a_0)^{-4}, \qquad a(t) = (t/t_0)^{1/2}, \qquad H = 1/2t.$$

Empty ($k = -1$)

$$\epsilon = 0, \qquad a(t) = (t/t_0), \qquad H = 1/t.$$

Cosmological constant, critical density ($k = 0$)

$$\epsilon = \epsilon_\Lambda = \frac{3H_0^2c^2}{8\pi G}, \qquad a(t) = e^{H_0(t-t_0)}.$$

For the $H(t)$ relation in the matter dominated case, note that if $a \propto t^{2/3}$ then

$$\frac{d\ln a}{d\ln t} = \frac{t}{a}\frac{da}{dt} = tH = \frac{2}{3}$$

and thus $H = 2/3t$. The same argument works for the radiation-dominated and empty cases.

Matter-dominated solutions with curvature are more complicated, but the solutions can be expressed in parametric form (textbook equations 6.17 and 6.18 for $k = +1$, and 6.20 and 6.21 for $k = -1$).

For $k = +1$, the universe expands, reaches a maximum at

$$a_{\rm max} = \frac{\Omega_0}{\Omega_0 - 1},$$

then recollapses and ends in a "big crunch."

For $k = -1$, the universe expands forever.
From the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2}\frac{1}{a^2(t)},$$

we can see that the two terms on the right-hand side are equal when $\Omega_0 = 0.5$.
At much earlier times, the first term must dominate, and the universe should evolve like a critical density universe.
At much later times, the second term must dominate, and the universe should evolve like an empty universe.

Specifically, when $a \ll \Omega_0/(1-\Omega_0)$, the expansion is very close to $a(t) \propto t^{2/3}$.
When $a \ll \Omega_0/(1-\Omega_0)$, the expansion is very close to $a(t) \propto t$, i.e., "free expansion" (not accelerating or decelerating).

A flat universe with matter and a cosmological constant

A flat universe dominated by matter and a cosmological constant appears to be a good description of the cosmos we live in.
The behavior is analogous to that of the $k = -1$ matter-dominated solution discussed above, except that the transition is to exponential expansion rather than free expansion.

The Friedmann equation in this case can be written

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \Omega_{\Lambda,0}\right].$$

The transition occurs at an expansion factor

$$\left(\frac{a}{a_0}\right) \sim \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}.$$

Note that for a critical density universe, we must have $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$.

Matter-radiation equality

The energy density of radiation in the universe is $\epsilon_{r,0} = a_{\rm SB}T^4$, where $T = 2.7\,{\rm K}$ is the temperature of the cosmic microwave background (CMB).

Roughly, CMB photons have $\lambda \sim 1\,\mathrm{mm}$, $kT \sim 10^{-3}\mathrm{eV}$, $n_\gamma \sim 10^3\,\mathrm{photons\,cm^{-3}}$, implying $\epsilon_{r,0} \sim 1\mathrm{eV\,cm^{-3}}$.
The contribution of starlight is negligible compared to the CMB.

The mean density of hydrogen atoms is roughly one atom per cubic meter. Hence $\epsilon_{\mathrm{bary},0} \sim 10^9\mathrm{eV}/10^6\,\mathrm{cm}^3 \sim 10^3\mathrm{eV\,cm^{-3}} \sim 10^3\epsilon_{r,0}$.

Since $\epsilon_{\mathrm{bary}} = \epsilon_{\mathrm{bary},0}(a_0/a)^3 = \epsilon_{\mathrm{bary},0}(1+z)^3$, and $\epsilon_r = \epsilon_{r,0}(a_0/a)^4 = \epsilon_{r,0}(1+z)^4$, the radiation and baryon densities were equal at

$$(1+z) \sim \frac{\epsilon_{\mathrm{bary},0}}{\epsilon_{r,0}} \sim 1000.$$

To be more precise, we need to use precise numbers for the above *and* include two other important contributions.
The radiation component includes neutrinos, which are nearly as numerous as CMB photons and are highly relativistic in the early universe.
The matter component includes dark matter, which appears to outweigh baryons by a factor $\sim 6:1$.

For $H_0 = 70\mathrm{km\ s^{-1}\ Mpc^{-1}}$,

$$\begin{aligned}
&\Omega_{\gamma,0} = 5.0 \times 10^{-5}\\
&\Omega_{\nu,0} = 3.4 \times 10^{-5}\\
&\Omega_{r,0} = \Omega_{\gamma,0} + \Omega_{\nu,0} = 8.4 \times 10^{-5}\\
&\Omega_{\mathrm{bary},0} = 0.04\\
&\Omega_{\mathrm{DM},0} \approx 0.26\\
&\Omega_{m,0} = \Omega_{\mathrm{bary},0} + \Omega_{\mathrm{DM},0} \approx 0.3.
\end{aligned}$$

These are the numbers in the "benchmark model" described in §6.5 of the textbook. More recent observational data lead to slightly different best estimates of these values, but not by much.

The value of $\Omega_{\gamma,0}$ is precisely known from the CMB temperature.
The cosmic neutrino background cannot be measured directly, but the value of $\Omega_{\nu,0}$ can be precisely calculated from theory given the standard model of particle physics. However, we have here treated neutrinos as *massless*, which is an excellent approximation in the early universe but not today.
The value of $\Omega_{\mathrm{bary},0}$ is well determined (at the $\approx 10\%$ level) by measurements of the cosmic deuterium abundance and by measurements of anisotropy in the CMB.
The value of $\Omega_{\mathrm{DM},0}$ is the most uncertain, but a variety of measurements imply it is probably known to 20%.

Based on these numbers, we conclude that radiation and matter had equal energy density at redshift

$$(1+z_{\rm eq}) = \frac{\Omega_{m,0}}{\Omega_{r,0}} \approx 3570.$$

At redshifts much higher than $z_{\rm eq}$, the universe was radiation dominated, with $a(t) \propto t^{1/2}$.
At $z_{\rm eq}$ there was a transition to a matter dominated universe, with $a(t) \propto t^{2/3}$.
At much lower redshift ($z \lesssim 2$), dark energy became important.
The age of the universe at $z_{\rm eq}$ is $t_{\rm eq} = 4.7 \times 10^4$ yrs.

The expansion history and cosmological parameters

Let's try to put all of this in more general terms.
The Friedmann equation is

$$H^2(t) = \frac{8\pi G}{3}\frac{\epsilon(t)}{c^2} - \frac{kc^2}{R_0^2 a^2(t)}.$$

We earlier showed (in our discussion of critical density, see also equation 4.31 of the textbook) that

$$\frac{kc^2}{R_0^2} = H_0^2(\Omega_0 - 1),$$

where $\Omega_0 = \epsilon_0/\epsilon_{c,0}$.
We can therefore recast the Friedmann equation in the form

$$\frac{H^2(t)}{H_0^2} = \frac{\epsilon(t)}{\epsilon_{c,0}} + \frac{1-\Omega_0}{a^2(t)}.$$

The value of $\epsilon(t)$ can be expressed in terms of the present day values of the matter and radiation densities and the value of the cosmological constant energy.

$$\epsilon(t) = \epsilon_{r,0}a^{-4} + \epsilon_{m,0}a^{-3} + \epsilon_\Lambda.$$

With the definition $\epsilon_{x,0} = \Omega_{x,0}\epsilon_{c,0}$, we have

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{(1-\Omega_0)}{a^2}.$$

The $\Omega_{\Lambda,0}$ term could be adjusted to allow for dark energy that varies in time.

This version of the Friedmann equation is one of the most useful for practical calculations.
For forms of dark energy that are not a cosmological constant, one needs to make the substitution

$$\Omega_{\Lambda,0} \longrightarrow \Omega_{\rm DE,0}\frac{\rho_{\rm DE}(z)}{\rho_{\rm DE,0}} = \Omega_{\rm DE,0}(1+z)^{3(1+w)},$$

where the last equality holds for constant w.

Note that throughout this discussion I have adopted our standard convention that the expansion parameter at the present day is $a_0 = 1$, and therefore $a \equiv (1+z)^{-1}$.

Luminosity distance

If we ignore the effects of curved space, cosmological expansion, and so forth, the relation between flux, luminosity, and distance is

$$f = \frac{L}{4\pi d^2}.$$

From the metric

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_k^2(r) d\Omega^2\right]$$

we can see that photons emitted isotropically by a source at comoving distance r are today (at t_0, with $a_0 = 1$) spread over a sphere of proper surface area

$$A_p(t_0) = 4\pi S_k^2(r),$$

since the solid angle is $\int d\Omega^2 = 4\pi$ steradians.

Recall that the comoving distance is defined so that it is equal to the proper distance at $t = t_0$, so for a flat universe with $S_k(r) = r$ this is just the usual Euclidean result.

For a positively curved space, the photons are spread over a smaller surface area than they would be in a flat universe, and for a negatively curved space they are spread over a larger surface area.

You might expect that the relation between flux, luminosity, and distance in an FRW universe would be

$$f = \frac{L}{4\pi S_k^2(r)} \qquad \text{(Warning: Incorrect Equation)}$$

However, there are two additional effects.

First, the energy of photons drops by a factor of $(1+z)$, and since f is an *energy* flux not a photon number flux, this reduces f by a factor of $(1+z)$.

Second, there is time dilation between the emitted frame and observed frame, so that photons emitted in a time interval Δt are received in a time interval $\Delta t(1+z)$. This reduces the flux by an additional factor of $(1+z)$.

Bottom line: The relation between flux, luminosity, and distance is

$$f = \frac{L}{4\pi S_k^2(r)(1+z)^2}.$$

The quantity $d_L = S_k(r)(1+z)$ is often called the *luminosity distance*, because

$$f = \frac{L}{4\pi d_L^2}.$$

Cosmologists also often refer to the angular diameter distance d_A, for which the angular size of an object of physical length l is $\theta = l/d_A$.

If you find yourself in need of formulas for cosmological distance measures, a good general reference is Hogg (1999, arXiv:astro-ph/9905116).

Standard candles and cosmological parameters

Suppose we have a class of objects of known luminosity L.

For example, Type Ia supernovae, produced by exploding white dwarfs, appear to have nearly the same peak luminosity, once one calibrates out variations using the rate at which they decline after the peak.

If we measure the redshift of such an object, we can determine its luminosity distance *if* we specify the values of $\Omega_{m,0}$, $\Omega_{r,0}$, and $\Omega_{\Lambda,0}$.

Following a line of reasoning we have used before:

$$\frac{1}{a}\frac{da}{dt} = H \quad \Rightarrow \quad dt = \frac{1}{H}\frac{da}{a}.$$

For propagating photons

$$dr = \frac{c\,dt}{a(t)} = \frac{c}{H}\frac{da}{a^2} = \frac{c}{H_0}\frac{H_0}{H}\frac{da}{a^2}.$$

With our most recent version of the Friedmann equation, we can write the comoving distance as

$$r = \frac{c}{H_0}\int_{a_e}^{1} \frac{da}{a^2\left[\Omega_{r,0}/a^4 + \Omega_{m,0}/a^3 + \Omega_{\Lambda,0} + (1-\Omega_0)/a^2\right]^{1/2}},$$

with $\Omega_0 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}$.

One can also use $a \equiv (1+z)^{-1} \Rightarrow da = -dz(1+z)^{-2} = -a^2 dz$ to write this formula as

$$r = \frac{c}{H_0}\int_0^z \frac{dz'}{\left[\Omega_{r,0}(1+z')^4 + \Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0} + (1-\Omega_0)(1+z')^2\right]^{1/2}},$$

Note that if $z \ll z_{\rm eq} \sim 10^4$, the $\Omega_{r,0}(1+z')^4$ term is negligible compared to the $\Omega_{m,0}(1+z')^3$ term.

If we have standard candles observed at several different redshifts, we can use their fluxes and the relation

$$f = \frac{L}{4\pi S_k^2(r)(1+z)^2}$$

to pin down the values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$.

In practice, people set out to use this method with supernovae to measure the deceleration of the universe and determine $\Omega_{m,0}$ and instead found an accelerating universe and showed that $\Omega_{\Lambda,0} > 0$.

In principle, perfect measurements at several redshifts provide enough information to separately determine $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ (recall that $\Omega_0 = \Omega_{m,0} + \Omega_{\Lambda,0} + \Omega_{r,0}$ by definition and that curvature is determined by Ω_0).

In practice, supernova measurements mostly constrain a combination of parameters that is roughly $\Omega_{m,0} - \Omega_{\Lambda_0}$. (Higher matter density produces stronger deceleration, and one can compensate this by putting in higher vacuum energy density, which produces stronger acceleration.)

To separately determine the two parameters, one needs other measurements that are sensitive to a different combination.

The characteristic size of hot spots and cold spots on the CMB turns out to be most sensitive to $\Omega_{m,0} + \Omega_{\Lambda,0}$, which is nicely complementary to the supernova constraints.

Baryon Acoustic Oscillation (BAO) Surveys

In principle, one can also play this game with "standard rulers" and angular sizes.

However, it has been difficult to identify a class of standard rulers that one doesn't expect to evolve with redshift.

A good candidate for such a standard ruler has emerged recently: a characteristic scale in the clustering of galaxies that is imprinted by acoustic oscillations (sound waves) in the early universe.

This scale is difficult to measure precisely because one must map galaxy clustering over enormous volumes, but several teams are setting out to do that.

This measurement method underpins some of the most ambitious cosmological surveys planned for the next decade.

The underlying goal is not really to measure $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ more precisely but to test whether cosmic expansion really is described by General Relativity with a cosmological constant.

Alternatives are: $\rho_{\rm DE}$ changes with time, or GR breaks down on cosmological scales (and thus the Friedmann equation itself is incorrect).

Final Remark

It has been common practice for decades to summarize the impact of cosmological parameters in the quantity

$$q_0 \equiv -\left(\frac{\ddot{a}a}{\dot{a}^2}\right)_{t_0},$$

which is a dimensionless measure of the current deceleration rate of the universe.

Standard formulas for the luminosity or angular size distance are expressed in terms of q_0.

In my view, this practice is no longer useful, because there is no unique relation between the single parameter q_0 and the two parameters $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$, which are the quantities of physical interest.