

Problem Set 3
Due Wednesday, April 18

Question 1: Circles on a Sphere (40 points)

(This is Problem 3.3 from the textbook.)

Suppose that you are a two-dimensional being living on the surface of a sphere of radius R . Show that if you draw a circle of radius r , the circle's circumference will be

$$C = 2\pi R \sin(r/R).$$

Idealize the Earth as a perfect sphere of radius $R = 6731$ km. If you could measure distances with an error of ± 1 meter, how large a circle would you have to draw on the Earth's surface to convince yourself that the Earth is spherical rather than flat?

Question 2: The Milne Cosmology (60 points)

This is a cosmological model based on special relativity, first analyzed by Milne in the 1930s. It describes a universe of zero mass density, so it cannot be a realistic model of our universe, but it nonetheless illustrates some useful features of more realistic cosmological models.

For this problem, adopt units in which the speed of light is $c = 1$. Velocities that appear in the problem are therefore dimensionless, and the value of $v = 0.1$, say, corresponds to a velocity of $1/10$ the speed of light.

Consider a flat spacetime, with zero matter density. At some event BB in this spacetime, particles of zero mass are sprayed in all directions, with varying speeds, up to c . (If we were being careful to make everything physically consistent, we would say “nearly flat spacetime,” “nearly zero matter density,” “nearly zero mass,” and “approaching c .”) Because spacetime is flat, we can use special relativity to analyze it.

An observer traveling on one of these particles erects a global coordinate system (t, r, θ, ϕ) , where t is the time since the event BB as measured by this observer. The metric in this coordinate system is just the usual flat spacetime metric in spherical coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2 d\gamma^2,$$

where $d\gamma^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

In this coordinate system, a particle with speed v (relative to this observer) has reached a distance $r = vt$, so this observer finds that Hubble's law $v = Hr$ holds, with $H = 1/t$.

The drawing on the next page shows a spacetime diagram of the universe in this coordinate system, with the θ and ϕ dimensionse suppressed. Note that because $c = 1$, both the r and t axes have units of length. (Or you could think of one as being marked in light-years, the other in years.)

We now want to understand what this spacetime looks like to a set of “comoving” observers who are attached to the particles. Spacetime paths of some representative observers are shown on the diagram by dashed lines.

(a) Why are none of these paths tilted by more than 45° ?

(b) Suppose that each particle carries a clock, measuring its proper time τ , again measured since the event BB. Recall that in special relativity, the clock of a moving observer runs slow by a factor $\tau/t = \sqrt{1 - v^2}$.

What are the surfaces $\tau = \text{constant}$ in the (t, r, θ, ϕ) coordinate system? Give an equation that defines these surfaces, and draw what one looks like on (your own version of) a diagram like the one below.

(c) Introduce a new spatial coordinate ω defined by the equation $r/\tau = \sinh(\omega)$. Argue that (ω, θ, ϕ) are *comoving* spatial coordinates, i.e. that each particle maintains constant values of ω , θ , and ϕ .

(d) Argue that for two events with $d\gamma = 0$ (i.e., two events whose spatial locations lie along a radial ray),

$$-dt^2 + dr^2 = -d\tau^2 + dl^2,$$

where dl is the spatial separation measured by the comoving observers (i.e., the spatial separation in a $\tau = \text{const.}$ surface).

(e) Show that the separation in a $\tau = \text{const.}$ surface is

$$dl^2 = -dt^2 + dr^2 + r^2 d\gamma^2 = \tau^2 d\omega^2 + \tau^2 \sinh^2(\omega) d\gamma^2,$$

and therefore that the full spacetime metric in the comoving coordinate system is

$$ds^2 = -d\tau^2 + \tau^2 [d\omega^2 + \sinh^2(\omega) d\gamma^2].$$

[*Hints:* Note that $d(\sinh^{-1}(u)) = (1 + u^2)^{-1/2} du$ and derive an expression for $d\omega$ in terms of dr , r , and τ . Then show that $t dt = r dr$ in a constant- τ surface, and forge ahead.]

(f) What is the relation between this metric and the Friedmann-Robertson-Walker metric described in section 3.3 of the textbook? How can space appear curved to the comoving observers even though the matter density of the universe is zero?

Extra Credit: What initial velocity distribution $N(v)$ [number of particles with velocity in the range $v \rightarrow v + dv$] is required in order to make the universe homogeneous on surfaces of constant τ ?