

Astronomy 682
Problem Set 3
Due Wednesday, April 22, in class

Question 1: Circles on a Sphere (40 points)

(This is Problem 3.3 from the textbook.)

Suppose that you are a two-dimensional being living on the surface of a sphere of radius R . Show that if you draw a circle of radius r , the circle's circumference will be

$$C = 2\pi R \sin(r/R). \quad (1)$$

Idealize the Earth as a perfect sphere of radius $R = 6731$ km. If you could measure distances with an error of ± 1 meter, how large a circle would you have to draw on the Earth's surface to convince yourself that the Earth is spherical rather than flat?

Hint: After writing down the necessary condition, you may want to use the Taylor expansion of $\sin(r/R)$ to get an equation you can solve analytically. Alternatively, you can solve the equation numerically by trial and error.

Question 2: Angular Sizes in an Expanding Universe (40 points)

Recall that:

- The FRW metric is

$$ds^2 = -c^2 dt^2 + a^2(t) [dr^2 + S_k^2(r) d\Omega^2],$$

where

$$\begin{aligned} S_k(r) &= R_0 \sin(r/R_0), & k &= +1, \\ &= r, & k &= 0, \\ &= R_0 \sinh(r/R_0), & k &= -1. \end{aligned} \quad (2)$$

- The comoving distance to an object that emitted light at time t_e is

$$r = \int_{t_e}^{t_0} \frac{c dt}{a(t)} \quad (3)$$

- The cosmological redshift is $1 + z = \frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e}$, and by definition $a_0 = a(t_0) = 1$.

(a) Consider a universe that has $a(t) = (t/t_0)^{2/3}$, flat space geometry ($k = 0$), and $t_0 = 14$ Gyr (14 billion years). What is the comoving distance r to a galaxy at redshift $z = 3$? Express your answer in light years.

(b) Suppose this galaxy is oriented perpendicular to the line of sight and subtends an angle $d\Omega = 3.6$ arc-seconds (10^{-3} degrees). What is the galaxy's physical size, in light years?

(c) Consider instead a universe that has $a(t) = t/t_0$, negative space curvature ($k = -1$), a curvature radius $R_0 = ct_0$, and $t_0 = 14$ Gyr. What is the comoving distance r to a galaxy at $z = 3$?

(d) Suppose this galaxy is oriented perpendicular to the line of sight and subtends an angle $d\Omega = 3.6$ arc-seconds (10^{-3} degrees). What is the galaxy's physical size, in light years?

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(e) Suppose you observed an object (maybe a galaxy, maybe something else) of known physical size and measured its redshift, $z = 3$. How could you use the observed angular size of the object to decide which of the two models (the one in a/b or the one in c/d) better describes the real universe? In which kind of universe would the object have the larger angular size?

Question 3: Expansion of a Self-Gravitating Sphere (40 points)

Consider a homogeneous sphere of mass M that is expanding homologously, which means that the expansion velocity of a shell at radius r is $v = Hr$. With a little thought, you can convince yourself that a sphere expanding in this way retains constant density throughout its volume at any given time, though the density drops with time as the sphere expands. H itself may depend on time, but it is constant throughout the sphere at a given time, and you do not need to use H in your solution below in any case.

At time t_0 , the radius of the sphere is R_0 . Its radius as a function of time is $R(t)$.

(a) Argue, based on Newtonian mechanics, that $R(t)$ should obey the equation

$$\frac{1}{2}\dot{R}^2 = \frac{GM}{R} + U, \quad (4)$$

where U is a constant. What is the physical interpretation of U ?

(b) Show that $R(t) = R_0(t/t_0)^{2/3}$ is a solution to equation (4) with $U = 0$, provided that the *density* of the sphere at time t_0 is

$$\rho_0 = \frac{3}{8\pi G} \times \frac{4}{9t_0^2}. \quad (5)$$

(c) Suppose that $U \gg \frac{GM}{R}$. In this limit, what is the solution for $R(t)$, in terms of R_0 , t_0 , and t ?