1 Overview

Reading: Ryden pp. 1-2, Shu ch. 1

1.1 Fluid description and mean free paths

Levels of description: quantum, classical particle, fluid continuum

Fluid: Describe particle velocity as $\vec{v} = \vec{u} + \vec{w}$ where $\vec{u} \equiv \langle\vec{v}\rangle$ is mean velocity in local fluid element.
Discuss density, velocity, temperature of fluid elements.
Fluid elements move, but particles shift from one fluid element to another only slowly, by diffusion.

What is necessary condition for fluid description to be valid?
$\lambda \ll L$ where

$$\begin{aligned} \lambda &\equiv \text{mean free path [cm]} \\ L &\equiv \text{characteristic size of system [cm]} \end{aligned}$$

Mean free path is $\lambda = 1/(n\sigma)$

$$\begin{aligned} n &\equiv \text{number density [cm}^{-3}\text{]} \\ \sigma &\equiv \text{collision cross-section [cm}^2\text{]} \end{aligned}$$

Typical cross-section for neutral atoms is $\sigma \sim 10^{-15}\,\mathrm{cm}^2$ ($\sim 10(\pi r_B^2)$ where $r_B \approx 0.5 \times 10^{-8}$ cm is hydrogen Bohr radius).
For air in room, $n \sim 10^{19}\,\mathrm{cm}^{-3}$, $\lambda \sim 10^{-4}$ cm.

Electron mean free path:
In an ionized gas, the electron's effective interaction radius r_e is determined by

$$\frac{e^2}{r_e} \sim m_e v_e^2 \sim kT \tag{1}$$

$$\begin{aligned} e &\equiv && \text{electron charge} && = 4.80 \times 10^{-10}\,\mathrm{g}^{1/2}\,\mathrm{cm}^{3/2}\,\mathrm{s}^{-1} \\ m_e &\equiv && \text{electron mass} && = 9.11 \times 10^{-28}\,\mathrm{g} \\ v_e &\equiv && \text{electron thermal velocity} && = \left(\frac{kT}{m_e}\right)^{1/2} \approx 4.2 \times 10^7 \left(\frac{kT}{1\,\mathrm{eV}}\right)^{1/2} \mathrm{cm\,s^{-1}} \\ k &\equiv && \text{Boltzmann's constant} && = 1.38 \times 10^{-16}\,\mathrm{erg/K} = (11,600)^{-1}\,\mathrm{eV/K} \\ T &\equiv && \text{electron temperature} && = [\mathrm{K}] \end{aligned}$$

$$r_e \equiv \text{ effective interaction radius } \sim 1.5 \times 10^{-7} \left(\frac{kT}{1\,\mathrm{eV}}\right)^{-1} \mathrm{cm}$$

Why is equation (1) reasonable?
Electrostatic energy $\sim$ thermal energy $\Longrightarrow$ strong interaction.

Mean free path

$$\lambda \sim (n\pi r_e^2)^{-1} \sim \frac{1}{\pi n}\left(\frac{kT}{e^2}\right)^2 \sim 1.5 \times 10^{13}\left(\frac{n}{\mathrm{cm}^{-3}}\right)^{-1}\left(\frac{kT}{1\,\mathrm{eV}}\right)^2 \mathrm{cm}. \tag{2}$$

Proper calculation gives result shorter by $\sim (\ln\Lambda)^{-1}$ because of effects of distant interactions.
$\Lambda \sim L_D/R_e \sim 10N_e^{-1/2}$ where L_D is Debye length, N_e is mean number of electrons within r_e.

ISM phases:
Cold molecular clouds, $T \sim 10\,\mathrm{K}$, $n \gtrsim 300\,\mathrm{cm}^{-3}$, $\lambda \lesssim 3 \times 10^{12}\,\mathrm{cm} \sim 0.2\,\mathrm{AU}$
Cool atomic clouds, $T \sim 100\,\mathrm{K}$, $n \sim 30\,\mathrm{cm}^{-3}$, $\lambda \sim 2\,\mathrm{AU} \sim 10^{-5}\,\mathrm{pc}$
Warm ionized medium, $T \sim 10^4\,\mathrm{K}$, $n \sim 0.3\,\mathrm{cm}^{-3}$, $\lambda \sim 10^{-5}\,\mathrm{pc}$
Hot ionized medium, $T \sim 10^6\,\mathrm{K}$, $n \sim 3 \times 10^{-3}\,\mathrm{cm}^{-3}$, $\lambda \sim 10\,\mathrm{pc}$

What do you notice about T, n numbers?
Approximate pressure equilibrium at $P = nkT \sim 4 \times 10^{-13}\,\mathrm{dyne\,cm}^{-2}$.

We will also consider examples from the intergalactic medium, for which the hydrogen number density is

$$n_H = \Omega_b \rho_c X/m_p = 1.71 \times 10^{-7}\left(\frac{X}{0.76}\right)\left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{\rho}{\bar{\rho}}\right)(1+z)^3 \ \mathrm{cm}^{-3}. \tag{3}$$

with

$$\lambda \sim \left(\frac{30}{\ln\Lambda}\right)\left(\frac{\rho}{\bar{\rho}}\right)^{-1}(1+z)^{-3}\left(\frac{kT}{1\,\mathrm{eV}}\right)^2 \ \mathrm{pc} \tag{4}$$

for fully ionized plasma with $n_e \approx n_H$.

General conclusions:

Fluid description valid for gas in many astrophysical contexts.
Frequent collisions ensure that particles only diffuse slowly from one fluid element to another, and that local velocity distribution is approximately isotropic and Maxwellian.
Distinguished from stellar dynamics and plasma physics, where $\lambda \gg L$.
Fluid description allows treatment of problems as ones of
fluid dynamics: 3 configuration space + 1 time dimension.

In stellar dynamics, one can still talk about densities and thermal energies of "fluid elements," but particles (stars) move from one element to another as fast as the fluid elements move themselves.

Some ideas from fluid dynamics are still applicable, but a complete solution requires
kinetic theory: 6 phase space + 1 time dimension.
The fluid dynamics simplification is therefore very powerful when it is applicable.

1.2 Euler Equations

We can get a long way with the Euler equations, which describe the dynamics of an inviscid, ideal gas.

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) &= 0 \quad \Longrightarrow \quad \frac{\partial \rho}{\partial t} + \vec{u} \cdot \vec{\nabla} \rho = -\rho \vec{\nabla} \cdot \vec{u} \\
\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} &= -\frac{1}{\rho} \vec{\nabla} P + \vec{g} \\
\frac{\partial \epsilon}{\partial t} + \vec{u} \cdot \vec{\nabla} \epsilon &= -\frac{P}{\rho} \vec{\nabla} \cdot \vec{u}.
\end{aligned}$$

$$\begin{aligned}
\rho &\equiv \text{mass density } [\,\mathrm{g\,cm^{-3}}] \\
\vec{u} &\equiv \text{fluid velocity } [\,\mathrm{cm\,s^{-1}}] \\
P &\equiv \text{pressure } [\,\mathrm{dyne\,cm^{-2}} \text{ or } \mathrm{erg\,cm^{-3}}] \\
\vec{g} &\equiv \text{gravitational acceleration } [\,\mathrm{cm\,s^{-2}}] \\
\epsilon &\equiv \text{specific internal energy } [\,\mathrm{erg\,g^{-1}}]
\end{aligned}$$

What do these equations express?
Local mass, momentum, and energy conservation.

Assuming that $\vec{g}$ is known (e.g., calculated from the Poisson equation), we have 5 equations for 6 unknowns (ρ, $\vec{u}$, P, ϵ). How do we close the set?
Use "constitutive relations" (a.k.a. equation of state, though we'll encounter more when we get to viscosity).
For example, for a monatomic gas of particle mass m,

$$\epsilon = \frac{1}{2}\langle|\vec{w}|^2\rangle = \frac{3}{2}\frac{kT}{m} \tag{5}$$

$$P = \frac{1}{3}\rho\langle|\vec{w}|^2\rangle = \frac{2}{3}\rho\epsilon = nkT. \tag{6}$$

What kinds of solutions might we look for?
Fully dynamic solutions from specified initial conditions.
Static solutions: $\vec{u} = 0$. Only non-trivial equation is (21), $\vec{\nabla} P = \rho \vec{g}$, hydrostatic equilibrium.
Steady state: $\frac{\partial Q}{\partial t} = 0$ for all quantities Q.

Self-similar: with appropriately rescaled variables, $\frac{\partial \tilde{Q}}{\partial \tilde{t}} = 0$.

Consider a hydrostatic equilibrium solution: water sitting on oil.
What is wrong with it? Water is denser, solution is Rayleigh-Taylor unstable. Same instability is why water falls out of a glass.

What is a question we should always ask about solutions? Are they stable to small perturbations?
What instabilities can affect a homogeneous medium?
Jeans instability (gravitational clumping) – if not stabilized by pressure.
Thermal instability – if gas can cool and cooling relations have particular properties.

What gas dynamical processes are missing from the Euler equations?

From the momentum equation:
Viscous forces.
Examples: slipping fluid layers. Colliding gas clouds (viscosity creates shocks)
Electromagnetic accelerations (be cautious about just adding to $\vec{g}$).

From the energy equation:
Viscous heating (same examples as viscous forces).
Heat conduction (example: hot and cold gas in pressure equilibrium).
Radiative heating and cooling.

In many cases, one can ignore viscous processes except in the neighborhood of shocks. In some cases, one can ignore electromagnetic forces (and energies) and/or radiative cooling.

Overview of where we are going:
Outline derivation of Navier-Stokes and Euler equations from Boltzmann equation.
Constitutive relations: pressure and viscosity.
Hydrostatic equilibrium solutions.
Sound waves and gravitational instability.
Shocks.
Applications, including: blast waves, accretion disks, intergalactic medium, galaxy formation, star formation. Somewhere along the way: radiative cooling, turbulence.