II. Spacetime and the Friedmann-Roberston-Walker Metric

Reading: Chapter 2

Special Relativity

Postulates of theory:

1. There is no state of "absolute rest."

2. The speed of light in vacuum is constant, independent of state of motion of emitter.

(Point 2 could really be subsumed into point 1.)

Implies: Simultaneity of events and spatial separation of events depend on state of motion of observer.

Relation between coordinate systems x, y, z, t and x', y', z', t' of uniformly moving observers is described by Lorentz transformations.

For a reference frame moving at constant velocity v in $+x$ direction, the *Galilean* transformation is

$$\begin{aligned} t' &= t \\ x' &= (x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

The *Lorentz* transformation is

$$\begin{aligned} t' &= \gamma \left(t - \frac{vx}{c^2} \right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z, \end{aligned}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the *Lorentz factor.*

Derived by considering a spherical light wave emitted at $t = t' = 0$, for which

$x^2 + y^2 + z^2 = c^2t^2$ must imply $x'^2 + y'^2 + z'^2 = c^2t'^2$

if c is observer independent.

Observers in relative motion disagree on the spatial separation $\Delta l = \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]^{1/2}$ and the time separation Δt between the same pair of events.

But they agree on the "spacetime interval" $(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ between events.

Analogy: Stationary observers with rotated reference frames disagree on Δx, Δy but agree on $\Delta l = \left[(\Delta x)^2 + (\Delta y)^2\right]^{1/2}$.

The Equivalence Principle

"Special" relativity restricted to uniformly moving observers. Can it be generalized?

Newtonian gravity: $\mathbf{a} = \mathbf{F}/m, \qquad \mathbf{F} = -GMm\hat{\mathbf{r}}/r^2.$
Why is this unsatisfactory?
  Implicitly assumes infinite speed of signal propagation.
  Coincidental equality of inertial and gravitational mass.

Einstein, 1907: "The happiest thought of my life." If I fall off my roof, I feel no gravity.

Equivalence between uniform gravitational field and uniform acceleration of frame. True in mechanics. *Assume* exact equivalence, i.e., for electrodynamics as well.
Equivalence principle implies gravitational and inertial masses *must* be equal.
Allows extension of relativity to accelerating frames.
Implies that extension of relativity *must* involve gravity.

Third frame trick $\longrightarrow$ gravitational mass of electromagnetic energy, gravitational redshift and time dilation, bending of light (incorrect answer because ignores curvature of space)

Restatement of equivalence principle: In the coordinate system of a freely falling observer, special relativity always holds locally (to first order in separation). No gravity.
Over larger scales (second order in separation), gravity doesn't vanish in a freely falling frame — tidal effects. E.g., freely falling objects in an inhomogeneous gravitational field may accelerate towards or away from each other.

Summary of General Relativity

With aid of equivalence principle, can change relativity postulate from "There is no absolute rest frame" to "There is no absolute set of inertial frames."
More informally, "There is no absolute acceleration."
Uniform acceleration can be treated as uniform gravitational field; the two are *indistinguishable.*

A *geodesic path* is a path of shortest distance, e.g.,
In flat space, a straight line.
On a sphere, a great circle.

In relativity, a geodesic path is a path of shortest *spacetime interval.*
In flat spacetime, a straight line at constant velocity.
Freely falling particles move along these geodesics in flat spacetime.

GR description of gravity:
All freely falling particles follow geodesic paths in curved spacetime.
Distribution of matter (more generally, stress-energy) determines spacetime curvature.

Misner, Thorne, and Wheeler's catchy summary of GR:
Spacetime tells matter how to move. (Along geodesic paths.)
Matter tells spacetime how to curve. (Field equation.)

Compare to equivalent description of Newtonian gravity:
Gravitational force tells matter how to accelerate. ($F = ma$.)
Matter tells gravity how to exert force. ($F = GMm/r^2$.)

The "equivalence of inertial and gravitational mass" in Newton's description is not a coincidence but a *necessary consequence* of the assumption that all freely falling particles follow geodesic paths.

Space curvature and the spatial metric

On a flat, two-dimensional surface, the angles of a triangle satisfy

$$\alpha + \beta + \gamma = \pi.$$

The spatial separation dl between two nearby points is

$$dl^2 = dx^2 + dy^2,$$

or, in polar coordinates,

$$dl^2 = dr^2 + r^2 d\theta^2.$$

The total length l of a path can be found by integrating dl along the path.

On the surface of a sphere, the angles of a triangle add to

$$\alpha + \beta + \gamma = \pi + A/R^2,$$

where A is the area enclosed by the triangle and R is the radius of the sphere.
If r is the spatial distance along a great circle from the origin (e.g., the North Pole) and θ the azimuthal angle (e.g., the longitude), then the spatial separation of nearby points is

$$dl^2 = dr^2 + R^2 \sin^2(r/R) d\theta^2.$$

Analogously, on a negatively curved (saddle-like) surface of constant curvature, the angles of a triangle add to

$$\alpha + \beta + \gamma = \pi - A/R^2,$$

and the spatial separation is

$$dl^2 = dr^2 + R^2 \sinh^2(r/R) d\theta^2.$$

Formulas that relate coordinate separations to length separations are called *metrics.*

These results for two-dimensional surfaces can be naturally generalized three-dimensional spaces.

The metrics for flat, positively curved, and negatively curved spaces of constant curvature radius R, in "spherical" coordinates, are, respectively,

$$\begin{aligned}
dl^2 &= dr^2 + r^2 \left[d\theta^2 + \sin^2\theta d\phi^2\right] \\
dl^2 &= dr^2 + R^2\sin^2(r/R) \left[d\theta^2 + \sin^2\theta d\phi^2\right] \\
dl^2 &= dr^2 + R^2\sinh^2(r/R) \left[d\theta^2 + \sin^2\theta d\phi^2\right].
\end{aligned}$$

Positive curvature $\Longrightarrow$geodesics "accelerate" (in 2nd derivative sense) towards each other. Initially "parallel" geodesics converge.

Example: great circles on a sphere.

Zero curvature $\Longrightarrow$no geodesic "acceleration." Initially parallel geodesics stay parallel. Euclidean geometry.

Example: straight lines on a plane.

Negative curvature $\Longrightarrow$geodesics "accelerate" away from each other. Initially parallel geodesics diverge.

Example: geodesics on a saddle.

Spacetime metric

In GR, as in differential geometry, a fundamental role is played by the *metric tensor* $g_{\mu\nu}$.

The indices μ, ν run from 0 to 3, representing the time coordinate and three spatial coordinates.

The metric tensor is symmetric, so it has 10 independent components rather than 16.

Spacetime interval between two events separated by small dx^μ is

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu.$$

In special relativity,

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2,$$

so $g_{\mu\nu} = \mathrm{diag}(-c^2, 1, 1, 1)$.

This is called the "Minkowski metric."

Note that even in flat spacetime, we would change the metric if we went to, e.g., polar coordinates, though they would still describe the same underlying geometry.

ds^2 is independent of coordinate system, hence state of motion of observer.

$ds^2 < 0: \ |ds| \ =$ proper time measured by an observer passing through events

$ds^2 > 0: \ |ds| \ =$ distance measured by an observer who sees both events as simultaneous

Can integrate $s = \int ds$ to get interval along a path between widely separated events.

Observers with different coordinate systems (e.g., moving relative to each other in arbitrary ways) disagree on values of dx^μ, $g_{\mu\nu}$.
All agree on value of ds^2.
Note that $g_{\mu\nu}$ is, in general, a function of spacetime position.

GR: Effect of geometry on matter

In GR, the equation of motion for freely falling particles in a specified coordinate system is just the equation for geodesic paths.
In practice, this equation represents four 2nd-order differential equations that determine $x^\alpha(s)$, given initial position and 4-velocity.

The geodesic equation is the relativistic analog of the Newtonian equation $\mathbf{g} = -\vec{\nabla}\Phi$.
The metric $g_{\mu\nu}$ is the relativistic generalization of the gravitational potential.

GR: Effect of matter on geometry

Need an equation to tell how matter produces spacetime curvature, since to get motion of particles we need the metric $g_{\mu\nu}$.
Must regain Newtonian gravity in appropriate limit $\longrightarrow$ use Poisson's equation for guidance: $\nabla^2\Phi = 4\pi G\rho$.
We want: [curvature] = [mass-energy density]

Several lines of argument (all complex) lead to the **Einstein field equation**

$$G_{\mu\nu} = 8\pi G_{\rm Newton} T_{\mu\nu}.$$

$G_{\mu\nu}$ is the *Einstein tensor*, built from $g_{\mu\nu}$ and its derivatives up to second order. (Like $\nabla^2\Phi$.)
$T_{\mu\nu}$ is the *stress energy tensor*, the relativistic generalization of density.
For an ideal fluid at rest, $T_{\mu\nu} = {\rm diag}(\rho, p/c^2, p/c^2, p/c^2)$.
The constant $8\pi G_{\rm Newton}$, where $G_{\rm Newton}$ is Newton's gravitational constant, is determined by demanding correspondence to Newtonian gravity in the appropriate limit.

Solutions of the field equation

Note that $G_{\mu\nu} = 8\pi G_{\rm Newton} T_{\mu\nu}$ is a set of ten, second-order differential equations for the ten components of $g_{\mu\nu}$.
Second-order $\Longrightarrow$

boundary conditions matter
spacetime can be curved even where $T_{\mu\nu} = 0$
propagating wave solutions exist

Nonlinear $\Longrightarrow$hard to solve.

Some exact solutions, e.g.

$\mathbf{T} = 0$ everywhere $\longrightarrow$ flat spacetime, "Minkowski space"

Spherically symmetric, flat at ∞, point mass at $r = 0$ $\longrightarrow$ Schwarzschild solution

Generalization to include angular momentum $\longrightarrow$ Kerr solution

Homogeneous cosmologies, which we will study

In other cases, approximate, by considering small departures from an exact solution (perturbation theory).

Recall that the (Newtonian) gravitational potential Φ has units of velocity^2.

The "weak field" limit of GR corresponds to $\Phi \ll c^2$. Spacetime curvature is weak; photons travel on nearly straight paths.

The combination of the weak field limit and $v \ll c$ leads to the Newtonian limit, in which GR approaches Newtonian gravity.

The Newtonian Limit

In the Newtonian limit, only the 00 (time-time) component of the geodesic equation is non-trivial. It yields

$$\mathbf{g} = \frac{1}{2}\vec{\nabla} g_{00}$$

for the gravitational acceleration $\mathbf{g}$ of a freely falling particle.

Thus, g_{00} can be identified with -2Φ, where Φ is the Newtonian gravitational potential.

The 00 component of the Einstein field equation leads to

$$\nabla^2 g_{00} = 8\pi G_{\rm Newton}(\rho + 3p/c^2).$$

For a non-relativistic fluid, $p \ll \rho c^2$, and we get the equation of motion of a particle moving under the influence of a gravitational potential Φ that obeys Poisson's equation.

The $3p/c^2$ contribution implies that radiation (with $p = \rho c^2/3$) has a stronger gravitational pull than matter (for the same energy density), and that a fluid with $p < -\rho c^2/3$ can exert gravitational push.

It's no surprise that Newton didn't discover these corrections, because he was only considering bodies with $v \ll c$.

The additional gravitational effects of pressure in GR will have some crucial implications for cosmology.

Tests of GR

- yields Newtonian gravity in appropriate limit
- precision tests of equivalence principle
- precession of Mercury – the key from Einstein's point of view
- bending of light – historically important

- gravitational redshift
- higher-order solar system tests $\Longrightarrow$measured values of "post-Newtonian parameters" agree with GR predictions
- binary pulsars:

  gravity wave dissipation rate – very strong test
  precession of orbit in an external system
  gravitational time delay, effects up to $\sim (v/c)^3$

Other low precision tests: structure of dense stars, gravitational lensing

Despite these impressive tests, application to cosmology requires gigantic extrapolation in length and time scale.

Can't rest comfortably on empirical basis of small-scale tests.

Cosmological models based on GR are impressively successful, but they require two strange ingredients: dark matter and dark energy.

Existence of these ingredients could be an indication that GR is breaking down in some way on cosmological scales, though we will generally take the view that it is not.

The Friedmann-Robertson-Walker Metric

In spherical coordinates, the Minkowski metric (no spacetime curvature, special relativity) can be written

$$ds^2 = -c^2dt^2 + r^2d\Omega^2,$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

is the angular separation.

In 1917, Einstein introduced the first modern cosmological model, based on GR, in which the *spatial* metric is that of a 3-sphere:

$$dl^2 = dr^2 + R^2\sin^2(r/R)d\Omega^2.$$

Here R is the curvature radius of the 3-dimensional space, and r is distance from the origin.

In the coordinate frame of a freely falling observer, time is just proper time as measured by the observer, and the spacetime metric is

$$ds^2 = -c^2dt^2 + dl^2.$$

This spacetime metric describes a homogeneous, isotropic, and static (unchanging) universe.

A natural generalization of the Einstein model is to allow the curvature radius to be a function of time.

The universe is still homogeneous and isotropic on a surface of constant t, but it is no longer static.

In the 1930s, Robertson and Walker (independently) showed that there are only three possible spacetime metrics for a universe that is homogeneous and isotropic.
They can be written

$$ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + S_k^2(r)d\Omega^2\right],$$

where

$$\begin{aligned} S_k(r) &= R_0 \sin(r/R_0), \qquad k = +1, \\ &= r, \qquad k = 0, \\ &= R_0 \sinh(r/R_0), \qquad k = -1. \end{aligned}$$

In this notation (the same as that used by Ryden), $a(t)$ is dimensionless.
It is defined so that $a(t_0) = 1$ at the time t_0 (usually taken to be the present) when the curvature radius is R_0. At other times the curvature radius is $a(t)R_0$.
The radial coordinate r, the radius of curvature R_0, and $S_k(r)$ all have units of length (e.g., Mpc).

At a time t when the expansion factor is $a(t)$, the spatial metric for $k = 0$ is

$$dl^2 = d(ar)^2 + (ar)^2 d\Omega^2.$$

This resembles the spatial part of the Minkowski metric, but the physical scale associated with a given comoving radius r changes in proportion to $a(t)$.
For $k = +1$, the spatial metric is

$$dl^2 = d(ar)^2 + (aR_0)^2 \sin^2(ar/aR_0)d\Omega^2.$$

This resembles the spatial metric of the static Einstein model, but the physical scale and the curvature radius are proportional to $a(t)$.

Friedmann and LeMaître used this metric in their cosmological models of the 1920s. Robertson and Walker proved that they are the only forms consistent with the Cosmological Principle (homogeneity and isotropy).
It is commonly called the Friedmann-Robertson-Walker (FRW) metric, or sometimes the Robertson-Walker metric.

Comoving Observers

The metric depends on the coordinate frame of the observer.
Even a homogeneous and isotropic universe only appears so to a special set of freely falling observers, called *comoving observers*.
These observers are "going with the flow" of the expanding universe. They have constant values of r, θ, and ϕ, and the proper distance between them increases in proportion to $a(t)$.

In the coordinate frame of these observers, the FRW metric applies, and the time coordinate of the FRW metric is just proper time as measured by these observers.

An observer moving relative to the local comoving observers has a "peculiar velocity," where peculiar is used in the sense of "specific to itself" rather than "odd."

An observer with a non-zero peculiar velocity does not see an isotropic universe – e.g., dipole anisotropy of the cosmic microwave background caused by reflex of the peculiar velocity.

Space curvature, and spacetime curvature

For $k = +1$, the space geometry at constant time is that of a 3-sphere, positively curved. The total volume of the universe is finite, though it grows in proportion to $a^3(t)$.

For $k = 0$, the space geometry at constant time is Euclidean, a.k.a. "flat space." Space is infinite.

For $k = -1$, the space geometry at constant time is that of a negatively curved, 3-dimensional "pseudo-sphere." Space is infinite.

Note: these are descriptions of *space* at constant t.

For many forms of $a(t)$, *spacetime* is positively curved even if *space* is not, (this is always the case unless a cosmological constant or some other form of energy with negative pressure is important).

In the special relativistic, Milne cosmology of Problem Set 3, spacetime is flat, but surfaces of constant time are negatively curved.

The substitution $x = S_k(r)$ allows the FRW metric to be written in another frequently used form:

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dx^2}{1 - kx^2/R_0^2} + x^2d\gamma^2\right].$$

because $du = S_k'(r)dr = (1 - kS_k^2(r))^{1/2}d\omega = (1 - ku^2)^{1/2}dr$ for $k = +1, 0, -1$.

With r as radial coordinate, radial distances are "Euclidean" but angular distances are not (unless $k = 0$). With x as radial coordinate, the reverse is true.

Proper distance and the Hubble parameter

In a curved, expanding universe, the idea of "distance" becomes complex. There are several reasonable and useful definitions of the distance to an object, or the distance between two objects.

The most straightforward of these definitions is the "proper distance" d_p, the distance to an object as measured in a surface of constant time.

If we follow a radial ray ($d\Omega = 0$) to an object with comoving radial coordinate r, the metric tells us that $dl = a(t)dr$ and thus

$$d_p = \int_0^r a(t)dr = a(t)r.$$

The value of r is the *comoving distance* to the object. The comoving distance doesn't change as the universe expands. With our definition that $a(t_0) = 1$, the comoving distance is equal to the proper distance at the present day.

The proper distance changes with time at a rate

$$\dot{d}_p = \dot{a}r = \frac{\dot{a}}{a} d_p.$$

Thus, the expanding universe leads to Hubble's law, $v = Hd$, with the Hubble parameter

$$H = \frac{\dot{d}_p}{d_p} = \frac{\dot{a}}{a}.$$

This is an important equation, relating the empirical parameter H discovered by Hubble to the expansion parameter of the Friedmann equation.

I've written H instead of H_0, because this identification of the Hubble parameter with $\dot{a}/a$ holds at any time, not just the present.

Light travel distance

We can also talk about the "light travel distance" to an object that emitted its light at a time t_e.

This is just $c(t_0 - t_e)$, where t_0 is the present age of the universe, since light always travels at c.

In GR, as in special relativity, light travels along "null" geodesics, with $ds = 0$.

For a radial ray ($d\Omega = 0$), the metric implies

$$cdt = a(t)dr,$$

which makes sense because $a(t)dr$ is the physical (proper) distance corresponding to coordinate separation dr at time t.

We can integrate this equation to get the proper distance to an object with light travel distance $c(t_0 - t_e)$:

$$d_p = a(t_0)r = a(t_0) \int_{t_e}^{t_0} dr = a(t_0) \int_{t_e}^{t_0} \frac{c\,dt}{a(t)}.$$

In an expanding universe, $a(t) \leq a(t_0)$, and the proper distance is always larger than the light travel distance because the expansion of the universe has "stretched" spatial scales during the time that the light was traveling.

We'll come later to notions of distance that involve angular separations, with $d\Omega^2 \neq 0$.

Redshift of photons

The frequency ν of a photon emitted by a nearby comoving source at distance d is Doppler shifted by a fractional amount

$$\frac{d\nu}{\nu} = \frac{-v}{c} = \frac{-Hd}{c} = -Hdt,$$

where H is the Hubble parameter and the last equality follows because $d = c\,dt$.
Substituting $H = \dot{a}/a$ yields

$$\frac{d\nu}{\nu} = -\frac{\dot{a}\,dt}{a} = -\frac{da}{a} \implies d\ln\nu = -d\ln a.$$

Let the photon be emitted with frequency ν_e at time t_e and observed with frequency ν_o at time t_o.
Integrate to get

$$\frac{\nu_e}{\nu_o} = \frac{a_o}{a_e} = \frac{\lambda_o}{\lambda_e} \equiv (1+z), \qquad z = \text{redshift}.$$

Constant of integration fixed by demanding $\nu_o \longrightarrow \nu_e$ as $a_o \longrightarrow a_e$.

The wavelength of a freely propagating photon stretches in proportion to $a(t)$.
This result can also be derived by considering successive wave crests traveling along null geodesics, as discussed in the textbook, or by solving the GR equation for evolution of 4-momentum along null geodesic.
This result underpins most of observational cosmology: if we can measure the redshift of light from a source, we know the expansion factor $a(t)$ at the time the source emitted its light.

Since light waves don't "disappear," the frequency shift implies time dilation: our clocks run slow compared to those of the emitting object.
This is a real effect. For example, if we observe distant (substantially redshifted) supernovae, they appear to rise and fall more slowly than nearby supernovae.
In the rest frame of the supernova, the rise and fall is the same as it is nearby, but in our frame the time has been stretched.