6 Sound Waves and Gravitational Instability

Reading: Ryden pp. 20-25, Shu pp. 110-112

6.1 Perturbation equations for a uniform medium

Consider the plane-parallel case $\Longrightarrow$ no y or z dependence

$$
\begin{aligned}
\text{Continuity:} \quad & \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \\
\text{Momentum:} \quad & \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g \\
\text{Poisson:} \quad & \frac{\partial g}{\partial x} = -4\pi G\rho.
\end{aligned}
$$

Now assume a uniform static medium with $\rho = \rho_0$, $P = P_0$, $u = 0$.
Symmetry $\Longrightarrow g = 0$, but Poisson's equation allows this only if $\rho = 0$.
We will accept the "Jeans swindle" and ignore this problem with our zeroth-order solution, jumping immediately to the perturbed equations.

Introduce small perturbations ρ_1, u_1, P_1, g_1

$$
\rho = \rho_0 + \rho_1(x,t) \qquad P = P_0 + P_1(x,t) \qquad u = u_1(x,t) \qquad g = g_1(x,t)
$$

where $|\rho_1/\rho_1| \ll 1$, $|P_1/P_0| \ll 1$.

Substitute these expressions and keep only the terms that are first-order in the perturbations:

$$
\begin{aligned}
\text{Continuity:} \quad & \frac{\partial \rho_1}{\partial t} + \rho_0\frac{\partial u_1}{\partial x} = 0 \\
\text{Momentum:} \quad & \frac{\partial u_1}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P_1}{\partial x} + g_1 \\
\text{Poisson:} \quad & \frac{\partial g_1}{\partial x} = -4\pi G\rho_1.
\end{aligned}
$$

Assume that $P = P(\rho)$, allowing the second equation to be written

$$
\rho_0\frac{\partial u_1}{\partial t} + \frac{\mathrm{d}P}{\mathrm{d}\rho}\Big|_0 \frac{\partial \rho_1}{\partial x} = g_1\rho_0.
$$

Note that this is standard first-order perturbation theory: introduce small perturbations, approximate full equations by keeping only terms first-order in the perturbations, subtracting off or otherwise using the zeroth-order equations as necessary.

Take the time derivative of the continuity equation, subtract the spatial derivative of the momentum equation and substitute from the Poisson equation

$$
\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0\frac{\partial^2 u_1}{\partial x\partial t} - \rho_0\frac{\partial^2 u_1}{\partial x\partial t} - \frac{\mathrm{d}P}{\mathrm{d}\rho}\Big|_0 \frac{\partial^2 \rho_1}{\partial x^2} = -\rho_0\frac{\partial g_1}{\partial x} = 4\pi G\rho_0\rho_1.
$$

Hence

$$\frac{\partial^2 \rho_1}{\partial t^2} - \frac{\mathrm{d}P}{\mathrm{d}\rho}\Big|_0 \frac{\partial^2 \rho_1}{\partial x^2} = 4\pi G \rho_0 \rho_1. \tag{40}$$

6.2 Negligible self-gravity: sound waves

$$\frac{\partial^2 \rho_1}{\partial t^2} - a^2 \frac{\partial^2 \rho_1}{\partial x^2} = 0, \qquad a = \left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\Big|_0\right)^{1/2}$$

is a wave equation for waves propagating at the sound speed a.
If $\rho_1(x,t) = f(x - at)$ where f is an arbitrary function and f' and f'' are its first two derivatives with respect to its argument:

$$\frac{\partial \rho_1}{\partial t} = -af', \qquad \frac{\partial^2 \rho_1}{\partial t^2} = a^2 f'', \qquad \frac{\partial^2 \rho_1}{\partial x^2} = f''$$

$$\Longrightarrow \frac{\partial^2 \rho_1}{\partial t^2} - a^2 \frac{\partial^2 \rho_1}{\partial x^2} = 0.$$

For $P = P_0(\rho/\rho_0)^\gamma$, $a_0 = \left(\frac{\gamma P_0}{\rho_0}\right)^{1/2} = \left(\frac{\gamma k T_0}{\mu m_p}\right)^{1/2}$.

Note that the sound speed is $\sim$ particle thermal speed; it is not surprising to learn that this is the speed with which a disturbance can propagate. In physical units,

$$a = 1.63 \times 10^6 \left(\frac{\gamma}{5/3}\right)^{1/2} \left(\frac{\mu}{0.6}\right)^{-1/2} \left(\frac{kT_0}{1\,\mathrm{eV}}\right)^{1/2} \mathrm{cm\,s^{-1}} \sim 16\,\mathrm{km\,s^{-1}}.$$

A memorable number is $(kT/m_p)^{1/2} \approx 10\,\mathrm{km\,s^{-1}}$ at $kT = 1\,\mathrm{eV}$, $T = 11,600\,\mathrm{K}$.

A sound wave cannot be created with wavelength less than the mean free path λ, which is ~ 0.2 AU even in a dense molecular cloud. The frequency of such a wave is $\sim a/\lambda \sim 10^{-8}$ Hz.

Ryden (pp. 22-24) shows that the effects of viscosity and heat conduction are to damp sound waves (converting acoustic energy into thermal energy) with an attenuation length

$$L_1 \sim \frac{\Lambda^2}{\lambda}, \quad \Lambda = \text{sound wavelength}, \quad \lambda = \text{mean free path}.$$

$\Longrightarrow$ if $\Lambda \gg \lambda$, sound propagates many wavelengths.
For sound waves in air, $\Lambda \approx 300\,\mathrm{m\,s^{-1}}/300\,\mathrm{Hz} = 1\,\mathrm{m}$, $\lambda \sim 10^{-4}\,\mathrm{cm} = 10^{-6}\,\mathrm{m}$, $L_1 \sim 10^6\,\mathrm{m}$.

6.3 Non-negligible self-gravity

Go back to equation (40). Consider a perturbation of the form

$$\rho_1(x,t) = \exp[i(\omega t - kx)].$$

Because sine waves are a complete basis set, we can represent any perturbation at time t_0 as a superposition of such perturbations.
This is an especially useful approach in linear perturbation theory because the different modes do not interact to first order, so we can solve for the evolution of each one separately and add up the results.

$$\frac{\partial^2 \rho_1}{\partial t^2} = -\omega^2 \rho_1 \qquad \frac{\partial^2 \rho_1}{\partial x^2} = -k^2 \rho_1,$$

so equation (40) implies

$$\omega^2 = k^2 a_0^2 - 4\pi G \rho_0.$$

For $k > k_J \equiv (4\pi G\rho_0)^{1/2}/a_0$, ω is real, implying sound waves that oscillate and propagate without growing.
For $k < k_J$, ω is imaginary: long wavelength perturbations grow exponentially because of self-gravity.
For $k \ll k_J$, the growth timescale is $\sim (G\rho_0)^{-1/2} \sim t_{\rm dyn}$.

The Jeans wavelength is

$$\lambda_J = \frac{2\pi}{k_J} \sim a_0 (G\rho_0)^{-1/2}.$$

Thus, a perturbation is gravitationally unstable if a sound wave cannot cross it in a dynamical time. Shorter wavelength perturbations are stable because pressure gradients build up fast enough to counter gravity.
A general perturbation may decompose into short wavelength components, which do not grow, and long wavelength perturbations, which do.
In the non-linear regime, perturbations of different wavelengths influence each other.

The Jeans criterion can also be derived approximately on energetic grounds, by considering a spherical region whose density is increased by a factor $(1+\delta)$, with $\delta \ll 1$.
The perturbed gravitational potential energy is

$$\Delta W \sim \frac{-GM\Delta M}{R} \sim \frac{-GM(R^3\rho_0\delta)}{R}.$$

The perturbed thermal energy is

$$\Delta U \sim \delta M a_0^2.$$

Thus $\Delta W + \Delta U < 0$ if

$$\frac{GM(R^3\rho_0\delta)}{R} > \delta M a_0^2 \quad \Longrightarrow \quad R > a_0 (G\rho_0)^{-1/2}.$$

Large scale perturbations decrease their energy by growing in amplitude, but small scale perturbations increase their energy by growing in amplitude.

Numerical values:

$$\lambda_J \equiv \frac{2\pi}{k_J} = \frac{\pi^{1/2} a_0}{(G\rho_0)^{1/2}} = 3.6\,\mathrm{kpc} \times \left(\frac{\gamma}{5/3}\right)^{1/2} \left(\frac{0.6}{\mu}\right) \left(\frac{kT_0}{1\,\mathrm{eV}}\right)^{1/2} \left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)^{-1/2}$$

$$M_J \equiv \frac{4\pi}{3} \left(\frac{\lambda_J}{2}\right)^3 \rho_0 = \frac{\pi}{6} \lambda_J^3 \rho_0 = 3.7 \times 10^8 M_\odot \times \left(\frac{\gamma}{5/3}\right)^{3/2} \left(\frac{0.6}{\mu}\right)^2 \left(\frac{kT_0}{1\,\mathrm{eV}}\right)^{3/2} \left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)^{-1/2}$$

in a regime relevant to cosmology, or

$$\lambda_J = 0.025\,\mathrm{pc} \times \gamma^{1/2} \left(\frac{2}{\mu}\right) \left(\frac{T_0}{10\,\mathrm{K}}\right)^{1/2} \left(\frac{n}{10^6\,\mathrm{cm}^{-3}}\right)^{-1/2}$$

$$M_J = 0.4 M_\odot \times \gamma^{3/2} \left(\frac{2}{\mu}\right)^2 \left(\frac{T_0}{10\,\mathrm{K}}\right)^{3/2} \left(\frac{n}{10^6\,\mathrm{cm}^{-3}}\right)^{-1/2}$$

in a regime relevant for star formation.