7 Shocks

Reading: Ryden chs. 3 & 4, Shu chs. 15 & 16. For the enthusiasts, Shu chs. 13 & 14.

A good article for further reading: Shull & Draine, The physics of interstellar shock waves, in Interstellar processes; Proceedings of the Symposium, Grand Teton National Park, WY, July 1-7, 1986 (A88-14501 03-90). Dordrecht, D. Reidel Publishing Co., 1987, p. 283-319.

Consider the propagation of a finite amplitude sound wave.
The speed of propagation is higher at higher temperature, so the crest of the wave gradually overtakes the trough ($T \propto \rho^{\gamma-1}$).
When faster moving gas overtakes slower moving gas, we get a discontinuous change of density and velocity, a shock.
(NB: In fact, this steepening happens even for $\gamma = 1$ because of non-linearity of the equations.)

Shocks can also be produced by any supersonic compressive disturbance. This is the more common source of shocks in astrophysical situations.
What are places where astrophysical shocks occur?
Cloud-cloud collisions
HII regions expanding into neutral medium
Stellar wind encountering medium
Supernova or GRB blast wave (internal and external shocks)
Accretion onto compact objects: spherical or disk
Accretion onto hydrostatic intracluster medium

In general a shock wave is
• a pressure driven compressive disturbance
• propagating faster than the "signal speed" for compressive waves
• producing irreversible change in fluid state (increase of entropy)

In most cases a shock involves a "discontinuous" change of fluid properties over a scale $\sim \lambda$.

7.1 Shock jump conditions

Consider a propagating shock wave in the rest frame of the shock.
Unshocked gas approaches from the $+x$ direction moving faster than its sound speed and passes through the shock.
Pre-shock conditions: ρ_1, u_1, T_1.
Post-shock conditions: $\rho_2 > \rho_1$, $u_2 < u_1$, $T_2 > T_1$.

We would like to derive the relations (a.k.a. "jump conditions") between ρ_1, u_1, T_1 (or, equivalently, P_1) and ρ_2, u_2, T_2 (or P_2), for a steady-state, plane-parallel shock ($\vec{u}$ perpendicular to shock front and fluid properties depend only on distance to front).

Within the shock front (a.k.a. "transition layer"), viscous effects are important – they cause the shock in the first place. However, outside this layer, viscous effects are small on scales larger than the mean free path. We will derive conservation equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}Q(\rho, u, P) = 0 \implies Q(\rho, u, P) = \text{constant},$$

and although the quantities Q involve viscous terms, we can ignore these outside the shock zone and can therefore derive the jump conditions from equations that don't involve viscosity terms.

We start from the continuity equation and the momentum equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0, \quad \frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} = \vec{g} - \frac{1}{\rho}\vec{\nabla}P + \frac{1}{\rho}\vec{\nabla}\cdot \overleftrightarrow{\pi},$$

and from the thermal energy equation, which it is helpful to write in the form (Ryden 1-43)

$$\frac{\partial}{\partial t}(\rho\epsilon) + \vec{\nabla}\cdot(\rho\epsilon\vec{u}) = -P\vec{\nabla}\cdot\vec{u} - \vec{\nabla}\cdot\vec{F} + \Psi,$$

analogous to the continuity equation, and to supplement with a separate equation (Ryden 1-44) for conservation of kinetic energy

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho u^2) + \vec{\nabla}\cdot(\tfrac{1}{2}\rho u^2\vec{u}) = \rho\vec{u}\cdot\vec{g} - \vec{u}\cdot\vec{\nabla}P + \vec{u}\cdot(\vec{\nabla}\cdot \overleftrightarrow{\pi}).$$

We now assume steady-state, $\frac{\partial}{\partial t} = 0$, plane-parallel, $\frac{\partial}{\partial y} = 0$, $\frac{\partial}{\partial z} = 0$, $\frac{\partial}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}x}$, and ignore gravity and viscosity. These equations then become

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\rho u) &= 0 && (41)\\
u\frac{\mathrm{d}u}{\mathrm{d}x} &= -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x} && (42)\\
\frac{\mathrm{d}}{\mathrm{d}x}(\rho\epsilon u) &= -P\frac{\mathrm{d}u}{\mathrm{d}x} && (43)\\
\frac{\mathrm{d}}{\mathrm{d}x}(\tfrac{1}{2}\rho u^2 u) &= -u\frac{\mathrm{d}P}{\mathrm{d}x}. && (44)\\
& && (45)
\end{aligned}$$

Equation (41) immediately gives

$$\rho u = \text{constant} \implies \rho_1 u_1 = \rho_2 u_2.$$

Using

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho u^2) = 2\rho u\frac{\mathrm{d}u}{\mathrm{d}x} + u^2\frac{\mathrm{d}\rho}{\mathrm{d}x} = \rho u\frac{\mathrm{d}u}{\mathrm{d}x} + u\left(\rho\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}\rho}{\mathrm{d}x}\right) = \rho u\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}}{\mathrm{d}x}(\rho u) = \rho u\frac{\mathrm{d}u}{\mathrm{d}x}$$

allows the equation (42) to be written

$$\rho u \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}P}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\rho u^2 + P\right) = 0 \tag{46}$$

so

$$\rho u^2 + P = \text{constant} \implies \rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2. \tag{47}$$

If we had kept the viscosity terms in the derivation, equation (46) would instead have been

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho u^2 + P - \frac{4}{3}\mu\frac{\mathrm{d}u}{\mathrm{d}x}\right) = 0. \tag{48}$$

Within the transition zone, where μ and $\frac{\mathrm{d}u}{\mathrm{d}x}$ are non-zero, $\rho u^2 + P$ is not constant.
However, in the pre-shock and post-shock zones, μ and $\frac{\mathrm{d}u}{\mathrm{d}x}$ are negligible, so equation (47) holds. We could use equation (48) (together with our other equations and the constitutive relation for viscosity) to follow what happens within the transition zone. However, the fluid approximation itself breaks down within this region.

Adding together equations (43) and (44) gives

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}x}\left(u\left(\tfrac{1}{2}\rho u^2 + \rho\epsilon\right) + Pu\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\rho u\left(\tfrac{1}{2}u^2 + \epsilon + \frac{P}{\rho}\right)\right) \\
&= \left(\tfrac{1}{2}u^2 + \epsilon + \frac{P}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}x}(\rho u) + \rho u \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{2}u^2 + \epsilon + \frac{P}{\rho}\right).
\end{aligned}$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}(\rho u) = 0$ and $\rho u \neq 0$, this equation implies that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{2}u^2 + \epsilon + \frac{P}{\rho}\right) = 0 \implies \tfrac{1}{2}u^2 + \epsilon + \frac{P}{\rho} = \text{constant},$$

so

$$\tfrac{1}{2}u_1^2 + \epsilon_1 + \frac{P_1}{\rho_1} = \tfrac{1}{2}u_2^2 + \epsilon_2 + \frac{P_2}{\rho_2}.$$

If we had been more complete, the conserved quantity would also include viscosity and heat conduction terms, but once again, these are unimportant outside of the transition zone.

In summary, we have the *Rankine-Hugoniot jump conditions* for a plane-parallel shock:

$$\rho_1 u_1 = \rho_2 u_2 \tag{49}$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \tag{50}$$

$$\tfrac{1}{2}u_1^2 + \epsilon_1 + \frac{P_1}{\rho_1} = \tfrac{1}{2}u_2^2 + \epsilon_2 + \frac{P_2}{\rho_2}. \tag{51}$$

Even though the physics of the shock region may be complicated and varied, these conditions follow from conservation of mass, momentum, and energy alone. More precisely, the first follows

from mass conservation, the second from mass and momentum conservation, and the third from mass and energy conservation. If ρ_1, u_1, and P_1 are known, we have three equations for the three unknowns ρ_2, u_2, and P_2.
Using $\epsilon_i = \frac{1}{\gamma_i - 1}\frac{P_i}{\rho_i}$, the last of the jump conditions can be written

$$\tfrac{1}{2}u_1^2 + \frac{\gamma_1}{\gamma_1 - 1}\frac{P_1}{\rho_1} = \tfrac{1}{2}u_2^2 + \frac{\gamma_2}{\gamma_2 - 1}\frac{P_2}{\rho_2},$$

for a gas that has a polytropic equation of state.
Note that γ_2 may be different from γ_1 if, for example, the shock dissociates molecules, or raises the temperature so that previously inaccessible degrees of freedom become accessible.

7.2 The Mach Number

The dimensionless number that characterizes the strength of a shock is the Mach number, the ratio of the shock speed to the upstream sound speed:

$$M_1 \equiv \frac{u_1}{a_1} = \left(\frac{\rho_1 u_1^2}{\gamma P_1}\right)^{1/2}. \tag{52}$$

The factor in () can be viewed as a ratio of "ram pressure" to thermal pressure in the pre-shock gas, or as a ratio of kinetic energy density to thermal energy density.

In terms of the Mach number, the shock jump conditions are (Ryden eqs. 3-51)

$$\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2} \\ \frac{P_2}{P_1} &= \frac{\rho_2 k T_2/m}{\rho_1 k T_1/m} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}. \end{aligned}$$

Together these conditions imply

$$\frac{T_2}{T_1} = \frac{[(\gamma-1)M_1^2 + 2][2\gamma M_1^2 - (\gamma-1)]}{(\gamma+1)^2 M_1^2}.$$

A strong shock is one with $M_1 \gg 1$, yielding

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \approx \frac{\gamma+1}{\gamma-1} = 4 \tag{53}$$

$$P_2 \approx \frac{2\gamma}{\gamma+1}M_1^2 P_1 = \frac{2}{\gamma+1}\rho_1 u_1^2 = \frac{3}{4}\rho_1 u_1^2 \tag{54}$$

$$T_2 \approx \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}T_1 M_1^2 = \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{m}{k}u_1^2 = \frac{3}{16}\frac{m}{k}u_1^2, \tag{55}$$

where the last equalities are for $\gamma = 5/3$.

In the rest frame of a strong shock, with $\gamma = 5/3$, the post-shock kinetic energy is

$$\frac{1}{2}u_2^2 \approx \frac{1}{32}u_1^2,$$

and the post-shock thermal energy is

$$\frac{3}{2}\frac{kT_2}{m} \approx \frac{9}{32}u_1^2.$$

So roughly half of the pre-shock kinetic energy is converted to thermal energy. The total energy of the post-shock gas is lower (in the shock rest frame) because of the work done on the gas by viscosity and pressure in the shock.

A weak shock has $M_1 - 1 = \epsilon \ll 1$. In this limit

$$\frac{\rho_2}{\rho_1} \approx 1 + \frac{4}{\gamma+1}\epsilon = 1 + \frac{3}{2}\epsilon \tag{56}$$

$$\frac{P_2}{P_1} \approx 1 + \frac{4\gamma}{\gamma+1}\epsilon = 1 + \frac{5}{2}\epsilon \tag{57}$$

$$\frac{T_2}{T_1} \approx 1 + \frac{4(\gamma-1)}{\gamma+1}\epsilon = 1 + \epsilon, \tag{58}$$

where the last equalities are again for $\gamma = 5/3$.

The post-shock Mach number is

$$M_2 \equiv \frac{u_2}{a_2} = \frac{u_1}{a_1}\frac{u_2}{u_1}\frac{a_1}{a_2} = M_1\frac{u_2}{u_1}\left(\frac{T_1}{T_2}\right)^{1/2}.$$

In the weak shock limit, $M_2 \approx 1 - \epsilon$.
In the strong shock limit

$$M_2 \approx M_1\frac{(\gamma-1)}{(\gamma+1)}\left[\frac{(\gamma+1)^2}{2\gamma(\gamma-1)M_1^2}\right]^{1/2} = \left(\frac{\gamma-1}{2\gamma}\right)^{1/2} \approx 0.45.$$

A shock converts supersonic gas into denser, slower moving, higher pressure, subsonic gas. It increases the specific entropy of the gas by an amount

$$s_2 - s_1 = c_V \ln\left(\frac{P_2}{\rho_2^\gamma}\right) - c_V \ln\left(\frac{P_1}{\rho_1^\gamma}\right) = c_V \ln\left(\frac{P_2}{P_1}\right) - c_V\gamma \ln\left(\frac{\rho_2}{\rho_1}\right).$$

One can also write this in the form

$$s_2 - s_1 = c_P \ln\left(\frac{T_2}{T_1}\right) - \frac{k}{m}\ln\left(\frac{P_2}{P_1}\right),$$

which is Ryden's eq. (3-55), using $c_P = c_V + \frac{k}{m} = \gamma c_V$.

In another terminology, a shock shifts gas to a higher adiabat.
An adiabat is a locus of constant entropy ($T \propto \rho^{\gamma-1}$) in the density-temperature plane. Gas can move adiabatically along an adiabat, while changes in entropy move it from one adiabat to another.

7.3 Numerical hydrodynamics and artificial viscosity

The physical scale of a shock transition layer is $\sim \lambda$.
In most systems of astrophysical interest, the scale of the system is $L \gg \lambda$.
In numerical hydrodynamics, it is usually impractical to make an individual grid cell comparable to or smaller than λ (especially in more than one dimension), so the physical scale of a shock is generally much smaller than one grid cell.
Fortunately, we usually don't care about the region right around the shock, just about the influence of the shock on the physical state of the gas.
The relation between the physical properties of the upstream and downstream gas are determined by the shock jump conditions, and they follow from conservation laws, independent of the details of the shock.

"Shock capturing codes" work by searching for places where shocks are "likely" to occur (e.g., where the velocity field is developing a discontinuity) and imposing shock jump conditions over the scale of one or two grid cells.

An alternative method, introduced by von Neumann and Richtmeyer in the 1950s, is to use an "artificial viscosity" that is much larger than the true viscosity one would calculate based on the microscopic properties of the gas.
High viscosity makes shocks broader, so that they can be resolved over several grid cells.
Since the jump conditions follow from conservation laws, the pre- and post-shock gas should still have the correct relations even if the viscosity is not the true viscosity.
There is a fair amount of experimentation and black magic in artificial viscosity — tuning things to get the desired physical behavior in as many situations as possible.

7.4 Radiative shocks

Now suppose that the gas is able to radiate energy in a "radiative relaxation layer" (rrl) after the shock.
Immediately after the shock front the fluid properties are still denoted ρ_2, u_2, P_2, T_2, but they change in the radiative relaxation layer, settling to new values ρ_3, u_3, P_3, T_3 further downstream.
The rrl is always large compared to the shock front itself because many collisions are required to cool the gas (so that the size of the rrl is $\gg \lambda$).

Within the rrl, the temperature drops, and the gas is squeezed to higher density. Since $\rho u =$ constant, the velocity drops.
We can demonstrate this by returning to the steady-state equations (41)-(43) but including the cooling term

$$L(\rho, T) \equiv -\frac{(\Gamma - \Lambda)}{\rho} > 0, \qquad [L] = \mathrm{erg\,g^{-1}\,s^{-1}}.$$

The equations are

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho u) = 0$$

$$
\begin{aligned}
u\frac{\mathrm{d}u}{\mathrm{d}x} &= -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x} \\
\frac{\mathrm{d}}{\mathrm{d}x}(\rho\epsilon u) &= \rho u\frac{\mathrm{d}\epsilon}{\mathrm{d}x} = -P\frac{\mathrm{d}u}{\mathrm{d}x} - \rho L.
\end{aligned}
$$

Combining the last equation with

$$
\epsilon = \frac{1}{\gamma-1}\frac{P}{\rho} \implies \frac{\mathrm{d}\epsilon}{\mathrm{d}x} = \frac{1}{\gamma-1}\left[\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x} - \frac{P}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}x}\right]
$$

and

$$
\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}x} = -\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}
$$

(since $\rho \propto u^{-1}$) implies

$$
\frac{u}{\gamma-1}\left[\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{P}{u}\frac{\mathrm{d}u}{\mathrm{d}x}\right] + P\frac{\mathrm{d}u}{\mathrm{d}x} = -\rho L,
$$

and substituting $\frac{\mathrm{d}P}{\mathrm{d}x} = -\rho u\frac{\mathrm{d}u}{\mathrm{d}x}$ implies

$$
-\frac{\rho u^2}{\gamma-1}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\gamma}{\gamma-1}P\frac{\mathrm{d}u}{\mathrm{d}x} = -\rho L
$$

and thus

$$
\frac{a^2-u^2}{\gamma-1}\frac{\mathrm{d}u}{\mathrm{d}x} = -L
$$

with $a^2 = \gamma P/\rho$. Since the post-shock flow is sub-sonic ($u < a$) and $L > 0$, we conclude that $\frac{\mathrm{d}u}{\mathrm{d}x} < 0$, and mass conservation therefore implies that $\frac{\mathrm{d}\rho}{\mathrm{d}x} > 0$.

The mass conservation and momentum conservation jump conditions apply as before, since their derivation did not involve ϵ (which is the quantity affected by radiative cooling). Therefore

$$
\begin{aligned}
\rho_3 u_3 &= \rho_2 u_2 = \rho_1 u_1 \\
\rho_3 u_3^2 + P_3 &= \rho_2 u_2^2 + P_2 = \rho_1 u_1^2 + P_1.
\end{aligned}
$$

As u drops in the rrl, $\rho u^2 = (\rho u)u$ drops and the pressure must rise.
For a strong shock with $\gamma = 5/3$,

$$
M_2 = u_2\left(\frac{5}{3}\frac{P_2}{\rho_2}\right)^{-1/2} \approx 0.45,
$$

so $P_2 \approx 3\rho_2 u_2^2$ and only a small rise in pressure is required to satisfy the second jump condition even if u_3 drops to zero.
The important point is that the pressure does not go *down*, and therefore a drop in temperature must be accompanied by a proportional rise in density to maintain the pressure.
If there is a lot of post-shock cooling, then the density ratio ρ_3/ρ_1 can be very high.

This is probably the most important difference between a non-radiative shock and a radiative shock: a non-radiative shock can only increase the density by a factor of ~ 4, but a radiative shock can increase the density by a very large factor.

A specific interesting case is that of an "isothermal shock," where $T_3 = T_1$. (This becomes the third jump condition.)
This can arise if, for instance, $T_1 = T_3$ is a temperature where the cooling time becomes very long, or a temperature where heating and cooling processes balance.
Combined with the equation of state $P = \frac{\rho}{m} kT = \rho a_T^2$, where $a_T = (kT/m)^{1/2}$ is the "isothermal sound speed," the solution to the shock jump conditions is,

$$\begin{aligned} \frac{\rho_3}{\rho_1} &= \frac{u_1}{u_3} = \left(\frac{u_1}{a_T}\right)^2 \equiv M_T^2 \\ T_3 &= T_1. \end{aligned}$$

Since

$$\rho_3 u_3^2 + P_3 = \rho_1 u_1^2 \left(\frac{u_1}{a_T}\right)^2 \left(\frac{a_T}{u_1}\right)^4 + \rho_1 a_T^2 \left(\frac{u_1}{a_T}\right)^2 = \rho_1 a_T^2 + \rho_1 u_1^2 = P_1 + \rho_1 u_1^2,$$

this solution satisfies the momentum jump condition.

The fact that $\rho_3/\rho_1 = M_T^2$ means that the compression factor in an isothermal shock can be arbitrarily high.

7.5 Oblique shocks

If the fluid enters the shock at an angle other than $90°$, the shock jump conditions apply only to the perpendicular component $u_\perp$ of u_1.
The parallel component of u_1 is unchanged, so the fluid is "refracted" by the shock, changing its direction so that it is closer to parallel to the front.

7.6 Shocks with magnetic fields: a few comments

Hydromagnetic shocks are often important in the ISM.
For a shock with B parallel to the shock front:

$$B_1 u_1 = B_2 u_2,$$

i.e., the magnetic field lines are compressed by the same factor as the density.

The momentum jump condition becomes

$$\rho_1 u_1^2 + P_1 + \frac{B_1^2}{8\pi} = \rho_2 u_2^2 + P_2 + \frac{B_2^2}{8\pi},$$

i.e., there is an additional "magnetic pressure" term $B^2/8\pi$.
(The units of B^2 are $\mathrm{erg\,cm^{-3}} = \mathrm{dyne\,cm^{-2}}$.)

The energy jump condition becomes

$$\tfrac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} + \frac{B_1^2}{4\pi\rho_1} = \tfrac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} + \frac{B_2^2}{4\pi\rho_2}.$$

Coupling of ions to magnetic fields can allow a shock front to be much narrower than the mean free path λ for particle-particle interactions.

The situation is more complicated when B is not parallel to the shock front, and it can be much more complicated in multi-fluid shocks when electrons, ions, and neutrals are coupled but not perfectly coupled.
One important effect is that magnetosonic waves (a.k.a. Alfven waves) can "warn" upstream ions and electrons about an approaching density discontinuity.
There are many additional classes of solutions, some of them discussed in the Shull & Draine article.