Radiative Gas Dynamics
Problem Set 6: 1-d Hydro Code: Gravity
Due Friday, March 9

Add gravity to your 1-d hydro code with artificial viscosity (you should actually add it as an option that can be either on or off). Specifically, change the velocity equation to

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - \frac{1}{\rho}\frac{\partial P}{\partial x} + g. \tag{1}$$

The gravitational acceleration is determined by the fluctuating part of the density field via the Poisson equation:

$$\frac{\partial g}{\partial x} = -\frac{\partial^2 \phi}{\partial x^2} = -4\pi G(\rho - \bar{\rho}_{\rm box}), \tag{2}$$

where $\bar{\rho}_{\rm box} = (\sum_i \rho_i)/N$ is the average density in the box.

We will determine g by integrating from $x = 0$. There are other methods that involve solving for and differentiating the gravitational potential, but in this 1-d case a straightforward integration will suffice. If you think about it carefully, you will realize that there are some subtle issues related to where the calculated gravitational acceleration actually is relative to the mass and velocity — at the center of a cell or the edge of a cell. The following scheme appears to work fine, and I recommend adopting it:

$$\begin{aligned} g_1 &= 0.0 \\ g_i &= g_{i-1} - 4\pi\Delta x \left[\tfrac{1}{2}(\rho_{i-1} + \rho_i) - \bar{\rho}_{\rm box}\right], \quad i = 2, N. \end{aligned} \tag{3}$$

Put a loop to do this calculation of g just after the loop that calculates the pressure (again, you will need two such loops per timestep, one for the half step, and one for the full step). Remember to apply periodicity: $g_0 = g_N$, $g_{N+1} = g_1$. The mean density $\bar{\rho}_{\rm box}$ can be calculated just once at the beginning of the program, after the initial density field is assigned, since mass is conserved.

One disadvantage of the scheme (3) is that it does not ensure that the acceleration of the box is zero, only that the acceleration of the first cell is zero. Thus, momentum is not conserved. You can correct this problem by computing an offset $g_{\rm off} = \sum \rho_i g_i / \bar{\rho}_{\rm box}$ and subtracting it from all of the accelerations after doing the calculation in (3). We will use symmetric initial conditions that ensure that the acceleration of the first cell *is* zero, so for the calculations here it won't make a difference whether you include this offset in your computations or not.

Note that equation (3) has adopted units in which $G = 1$. What does this mean? Why didn't we adopt $G = 2$, or $G = 6.67 \times 10^{-8}$? The real way that G enters the calculation is by effectively defining the unit of density. In our previous calculations, the value of the density has been arbitrary, and only the fractional perturbation to the density mattered. (Try rerunning one of the sound wave perturbations with $\rho = 10$ instead of $\rho = 1$ and the same fractional perturbation amplitude — you'll get the same results except for the overall change of the density.) Now, however, the value of ρ will determine the importance of gravity relative to pressure gradients. Alternatively, one may think of gravity as introducing a new timescale to our calculations. Previously, the only characteristic timescale (and the one we used to define our time unit) was the sound crossing time of the box.

What new timescale does gravity introduce? How does the ratio of this timescale to the sound crossing time depend on the mean density $\bar{\rho}$? How does it depend on the temperature T?

For all of the the calculations below, return to the periodic boundary condition, instead of the hard boundary condition.

First use the Gaussian perturbation initial conditions adopted in PS 4, slightly modified because now we won't necessarily have the unperturbed density $\rho = 1$. The initial density field is

$$\rho(x, t=0) = \bar{\rho}\left(1 + \Delta \exp\left[-\frac{(x-0.5)^2}{2 \times 0.05^2}\right]\right), \tag{4}$$

the initial velocity field is $u(x) = 0.0$, and the initial temperature field is

$$T(x, t=0) = \bar{T}\left(\frac{\rho}{\bar{\rho}}\right)^{\gamma-1}. \tag{5}$$

(Note that it is now $\bar{T}$ that should be specified, not T_1, because we don't necessarily want to change the mean temperature if we change the mean density. Note also that there is a subtle difference between $\bar{\rho}$ and $\bar{\rho}_{\rm box}$, since the presence of the perturbation means that the average density in our box is not actually the unperturbed average density $\bar{\rho}$.)

Evolve these initial conditions using $\bar{\rho} = 1$, $\bar{T} = 1$, $\Delta = 0.02$, $N = 200$, $\Delta t = 2 \times 10^{-4}$, $C_q = 4$, $D_q = 0.1$. Plot the density field at $t = 0.1$, 0.3, 0.7, 0.8, 0.9, 1.0. How do the results compare (qualitatively) to the results you obtained in PS 4 with no gravity?

(If you get stuck and can't get your code to work, you can look at or, if necessary, use mine, which is at `http://www.astronomy.ohio-state.edu/~A825/hydro3.c`.)

Now evolve the same initial conditions but with $\bar{\rho} = 10$ (still $\bar{T} = 1$). You may not make it all the way to $t = 1.0$. Plot the results, and explain the difference from the case for $\bar{\rho} = 1$.

To understand the gravitational effects quantitatively, it is helpful to return to the standing wave initial conditions, which pick out a single wavelength. Modified to allow for a mean density other than 1, these are

$$\rho(x, t=0) = \bar{\rho}(1 + \Delta\cos(kx)) \qquad \text{for all } x, \tag{6}$$

with $k = 2\pi/\lambda$, $u = 0$ and $T(x, t=0)$ determined by equation (5).

For all of the remaining calculations, use $N = 800$ (to get better resolution of shocks), $\Delta t = 10^{-4}$, $C_q = 4$, $D_q = 0.5$. Evolve to $t = 2.0$ or until the program stops because of negative densities or thermal energies (or arithmetic errors). Obtain output at time intervals of 0.1, and plot the density, velocity, temperature, and entropy fields for $t = 0.1$, 0.3, and the last four output times before the calculation halts (e.g., if the calculation crashes at $t = 1.15$, plot $t = 0.1$, 0.3, 0.8, 0.9, 1.0, 1.1). Start with a perturbation amplitude $\Delta = 0.02$ in all cases. I have put my routines for making multi-panel plots that show all of these quantities at multiple output times on the course web page, if you want to use them.

Do the following cases:

- $\lambda = 0.5,\ \bar{\rho} = 25,\ \bar{T} = 1$
- $\lambda = 0.5,\ \bar{\rho} = 25,\ \bar{T} = 4$
- $\lambda = 0.5,\ \bar{\rho} = 15,\ \bar{T} = 1$
- $\lambda = 0.25,\ \bar{\rho} = 25,\ \bar{T} = 1$

Describe your results for the first case in moderate detail – i.e., explain whatever features of your numerical results you can.

You will probably get gravitational collapse eventually in all cases except the second, but the time evolution should be very different for the last two from the first — almost no growth in the amplitude of the perturbation for a long time, then rapid growth once non-linear changes to the density field push the system into instability. The clearest way to see this will be to look at the energy vs. time. Since your equation for energy (equation 6 of PS 4) does not include gravitational potential energy, it will not be conserved once gravity is included, and the departure from the initial energy provides a measure of the strength of the perturbation.

Why is the behavior of the first case fundamentally different from that of the subsequent three? (Hint: calculate the Jeans wavelength for $\bar{T} = 1,\ \bar{\rho} = 25$.)