# VIII. Linear Fluctuations

## Linear perturbation theory in general

Start with an "unperturbed" solution for quantities $x_i$, $i = 1, 2, ...$

Write equations for $\tilde{x}_i = x_i + \delta x_i$.

Expand, keeping terms linear in $\delta x_i$, and subtracting off unperturbed solution to get equations for $\delta x_i$.

Hope you can solve them.

In GR, the quantities $x_i$ might be metric coefficients, densities, velocities, etc. GR perturbation theory can be tricky for two reasons.

(1) The equations are very complicated. This is a problem of practice, not of principle.

(2) In a perturbed universe, the "natural" coordinate choice is no longer obvious. Different coordinate choices have different equations. Results for physically measurable quantities should be the same, but the descriptions can look quite different.

For scales $l \ll cH^{-1}$ and velocities $v \ll c$, one can usually get by with Newtonian linear perturbation theory.

## Newtonian perturbation theory in an expanding universe

(This discussion follows Peebles, pp. 111-119.)

*Basic unperturbed equations*

In an inertial frame, the equations governing the density $\rho(\mathbf{r}, t)$ and velocity $\mathbf{u}(\mathbf{r}, t)$ of an ideal fluid are:

$$\text{Continuity equation}: \quad \left(\frac{\partial \rho}{\partial t}\right)_{\mathbf{r}} + \vec{\nabla}_{\mathbf{r}} \cdot (\rho \mathbf{u}) = 0 \quad \text{(mass conservation)}$$

$$\text{Euler equation}: \quad \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{r}} + (\mathbf{u} \cdot \vec{\nabla}_{\mathbf{r}})\mathbf{u} = -\vec{\nabla}_{\mathbf{r}} \Phi \quad \text{(momentum conservation)}$$

$$\text{Poisson equation}: \quad \nabla^2_{\mathbf{r}} \Phi = 4\pi G \rho.$$

We have ignored pressure gradients in the Euler equation.

*Transformation of variables*

We would like to have these equations in comoving coordinates $\mathbf{x} = \mathbf{r}/a(t)$ with "peculiar" velocity $\mathbf{v} = \mathbf{u} - (\dot{a}/a)\mathbf{r} = \dot{(a\mathbf{x})} - \dot{a}\mathbf{x} = a\dot{\mathbf{x}}$.

We want to use a quantity that will be small when perturbations are small, so define the dimensionless density contrast $\delta(\mathbf{x}, t)$ by $\rho = \rho_b(t) [1 + \delta(\mathbf{x}, t)]$, $\rho_b$ = background density $\propto 1/a^3$.

By the chain rule, the time derivative of a function $f$ at fixed time $t$ and comoving position $\mathbf{x} = \mathbf{r}/a$ is

$$\left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{r}} + \left(\frac{\partial \mathbf{r}}{\partial t}\right)_{\mathbf{x}} \cdot \vec{\nabla}_{\mathbf{r}} f = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{r}} + \frac{\dot{a}}{a}\frac{\mathbf{r}}{a} \cdot \left(a\vec{\nabla}_{\mathbf{r}} f\right).$$

Rearranging terms yields the expression for the time derivative of $f$ at fixed $t$ and $\mathbf{r}$:

$$\left(\frac{\partial f}{\partial t}\right)_{\mathbf{r}} = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} - \frac{\dot{a}}{a}\mathbf{x} \cdot \vec{\nabla} f, \quad \vec{\nabla} \equiv a\vec{\nabla}_{\mathbf{r}}.$$

With appropriate substitutions, the three equations above become:

$$\begin{aligned} \text{Continuity :} \quad & \frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla} \cdot [(1+\delta)\mathbf{v}] = 0 \\ \text{Euler :} \quad & \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a}(\mathbf{v} \cdot \vec{\nabla})\mathbf{v} + \frac{\dot{a}}{a}\mathbf{v} = -\frac{1}{a}\vec{\nabla}\phi \\ \text{Poisson :} \quad & \nabla^2\phi = 4\pi G\rho_b a^2 \delta \quad \text{with} \quad \phi \equiv \Phi - \frac{2}{3}\pi G \rho_b a^2 x^2. \end{aligned}$$

The one qualitatively new feature is the "friction" term $\left(\frac{\dot{a}}{a}\right)\mathbf{v}$ in the Euler equation. It drags $\mathbf{v}$ to 0 if $\vec{\nabla}\phi$ vanishes.

This is just the "kinematic redshift" that we encountered long ago; it reflects the use of expanding, non-inertial coordinates.

It will have a crucial consequence: density contrasts in an expanding universe grow as power laws of time (for $\Omega_m = 1$) instead of exponentials.

*The linear perturbation equation*

So far, the equations are general, within the Newtonian limit and the assumption that pressure gradients are negligible, $\vec{\nabla}p/\rho \ll \vec{\nabla}\phi$.

The linear approximation is obtained by keeping only the terms that are first order in $\delta$ or $\mathbf{v}$:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla} \cdot \mathbf{v} = 0, \qquad \frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\vec{\nabla}\phi = 0.$$

Eliminate $\mathbf{v}$ by taking time derivative of first equation, $(1/a) \times$ divergence of second, subtracting, and substituting using the continuity and Poisson equations.

Result:

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = 4\pi G \rho_b \delta.$$

This is a second-order differential equation for $\delta(\mathbf{x}, t)$ with a growing and decaying mode solution:

$$\delta(\mathbf{x}, t) = A(\mathbf{x})D_1(t) + B(\mathbf{x})D_2(t).$$

*The linear growth factor*

We can rewrite the equation for $\delta$ as an equation for the growth factor

$$\ddot{D} + 2H(z)\dot{D} - \frac{3}{2}\Omega_{m,0}H_0^2(1+z)^3 D = 0.$$

For $\Omega_m = 1$, $D_1(t) \propto t^{2/3} \propto a(t)$, $D_2(t) \propto t^{-1} \propto a^{-3/2}$.

A more complicated algebraic expression for $D_1(t)$ in the case of $\Omega_m < 1$, $\Omega_\Lambda = 0$ is given in Peebles (1980, Large Scale Structure of the Universe), equation 11.16.

For a flat universe with a cosmological constant, the solution to the differential equation can be written in integral form

$$D_1(z) = \frac{H(z)}{H_0} \int_z^\infty \frac{dz'(1+z')}{H^3(z')} \left[ \int_0^\infty \frac{dz'(1+z')}{H^3(z')} \right]^{-1} ,$$

where the factor in brackets makes the normalization $D_1 = 1$ at $z = 0$.

For other equations of state of dark energy, or for more components (curvature, radiation), no general integral expression exists, and the differential equation must be solved directly.

To a good approximation, $d\ln D_1/d\ln a \approx \Omega_m^{0.6}$, implying $D_1 \propto a$ while $\Omega_m \approx 1$ and slowing of growth as $\Omega_m$ falls below one.

*The linear density and velocity fields*

If we assume that density fluctuations appear at some very early time, then at a much later time we care only about the growing mode $D_1(t)$, which we can just call $D(t)$. Thus,

$$\delta(\mathbf{x}, t) = \delta(\mathbf{x}, t_i)\frac{D(t)}{D(t_i)}, \qquad D(t) \propto a(t) \propto t^{2/3} \text{ for } \Omega_m = 1.$$

As long as $\delta \ll 1$, density contrasts simply “grow in place” in comoving coordinates.

From the continuity equation

$$\vec{\nabla} \cdot \mathbf{v} = -a\frac{\partial \delta}{\partial t} = -a\delta\frac{\dot{D}}{D} = -a\delta H f(\Omega_m),$$

where

$$f(\Omega_m) \equiv \frac{1}{H}\frac{\dot{D}}{D} = \frac{d\mathrm{ln}D}{d\mathrm{ln}a} \approx \Omega_m^{0.6}.$$

The condition for a perturbation to be growing mode is $\dot{\delta} = \delta H f(\Omega_m)$; growing mode perturbations are those for which the velocity divergence correctly reinforces the gravitational growth.

**Newtonian linear perturbation theory, bottom lines**

- For $\Omega_m = 1$, $\delta(\mathbf{x}, t) \propto a(t) \propto t^{2/3}$ [growing mode, matter dominated, no pressure].
- For $\Omega_m \neq 1$, $\delta(\mathbf{x}, t) = \delta(\mathbf{x}, t_i)D(t)/D(t_i)$, but $D(t)$ is more complicated. Generally $d\ln D/d\ln a = f(\Omega_m) \approx \Omega_m^{0.6}$, so fluctuation growth slows when $\Omega_m$ drops below 1.
- The peculiar velocity field satisfies $\vec{\nabla} \cdot \mathbf{v} = -a\delta H f(\Omega_m)$.
- Pressure gradients can stabilize fluctuations on scales smaller than $c_s(G\rho_b)^{-1/2}$, where $c_s$ is the sound speed.
- Because of kinematic redshift, linear perturbations grow roughly as power laws of time rather than exponentially, so they do not forget their initial conditions.
- All this changes when $|\delta|$ reaches $\sim 1$.

**Fourier description**

It is often convenient to work with Fourier components of $\delta$:

$$\delta(\mathbf{x}, t) = \int d^3k \delta_{\mathbf{k}} d^{2\pi i \mathbf{k}\cdot\mathbf{x}}, \qquad \delta_{\mathbf{k}} = \int d^3x \delta(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}.$$

Each $\delta_{\mathbf{k}} = A_{\mathbf{k}} e^{i\theta_{\mathbf{k}}}$ is a complex number, the Fourier amplitude. (Beware that different authors choose different conventions on where to put the $2\pi$'s.)

If the phases $\theta_{\mathbf{k}}$ are uncorrelated, then the field $\delta(\mathbf{x})$ is *Gaussian*. This implies that the 1-point probability distribution function of $\delta$ is Gaussian,

$$P(\delta) = (2\pi\sigma^2)^{-1/2} e^{-\delta^2/2\sigma^2},$$

where the variance

$$\sigma^2 \equiv \langle \delta^2 \rangle = \int_0^\infty 4\pi k^2 P(k) dk.$$

Here $P(k)$ is the *power spectrum*

$$P(k) \equiv \langle A_{\mathbf{k}} A_{\mathbf{k}}^* \rangle,$$

which tells the rms fluctuation as a function of scale (whether or not the field is Gaussian).

In linear theory, each Fourier mode evolves independently, $\delta_{\mathbf{k}}(t) \propto D(t)$.

The spatial scale is $\lambda = 2\pi/k$, so even if small scale modes have become nonlinear, $4\pi k^3 \delta_{\mathbf{k}} \gtrsim 1$, large scale modes may still follow linear theory. (This statement is not obviously true, but it holds in most circumstances.)

On scales in the linear regime, the shape of the power spectrum is preserved, and its amplitude grows $\propto D^2(t)$.

**Variance of mass fluctuations**

We are often interested in the properties of the density field after it has been smoothed, by convolution with a window function $W_R(r)$ of characteristic radius $R$.

Common choices for $W(r)$ include a top hat,

$$W(r) = \Theta\left[1 - \frac{r}{R}\right]$$

and a Gaussian

$$W(r) = e^{-r^2/2R^2}.$$

The variance of fluctuations of the smoothed density field is

$$\begin{aligned} \sigma^2(R) &= \int_0^\infty 4\pi k^2 P(k) dk\, \widetilde{W}^2(kR) \\ &= \int_0^\infty \Delta^2(k) d\ln k\, \widetilde{W}^2(kR), \\ \text{where } \Delta^2(k) &\equiv 4\pi k^3 P(k) \end{aligned}$$

and

$$\widetilde{W}(kR) = \int d^3r W_R(r) e^{-2\pi i \mathbf{k}\cdot\mathbf{r}}$$

is the Fourier transform of the window function.

If $W_R(r)$ depends only on $r/R$, then $\widetilde{W}(\mathbf{k})$ depends on $\mathbf{k}$ only through the combination $kR$, and it generally satisfies

$$\begin{aligned} \widetilde{W} &\approx 1 \qquad 2\pi kR \ll 1 \\ \widetilde{W} &\approx 0 \qquad 2\pi kR \gg 1. \end{aligned}$$

For a power spectrum $P(k) \propto k^n$ with $n > -3$, $\sigma^2(R) \propto R^{-(3+n)}$ increases towards smaller scales and is dominated by waves with $k \sim 1/(2\pi R)$.

These results hold generally, not just in linear theory, provided that one uses the non-linear $P(k)$ rather than the linear $P(k)$. However, if one starts with Gaussian fluctuations, the probability distribution $P(\delta)$ becomes non-Gaussian in the non-linear regime.

**The Inflation + Cold Dark Matter Power Spectrum**

*The scale-invariant power spectrum*

A scale that exits the Hubble radius during inflation with fluctuation amplitude $\delta_H$ reenters the Hubble radius during the radiation or matter-dominated era with approximately the same amplitude.

If the fluctuations are scale-invariant, then this amplitude $\delta_H$ is constant, but it applies at a different time for each scale.

To understand the implications of this, it is useful to blow up the upper right of the "last-in first-out" inflation diagram (see figures at end of section).

Consider fluctuations that reenter the Hubble radius during the matter-dominated era, when $D_1(a) \propto a$.

What does $\delta_H =$const. imply for the power spectrum $P(k,t)$ at a fixed time, where $k$ represents the *comoving* wavenumber $2\pi/\lambda$?

Scale-invariance implies

$$4\pi k^3 P(k,t_c) \approx \delta_H^2 \approx \text{const.},$$

where $t_c$ is the time when the comoving scale $\lambda$ "crosses" (is equal to) the Hubble radius.

The condition for crossing the Hubble radius is:

$$a(t_c)\lambda \sim ct_c \implies a(t_c)/k \propto a^{3/2}(t_c) \implies a^{1/2}(t_c) \propto 1/k,$$

where we have used $a \propto t^{2/3}$ for a matter dominated universe.

Since $P(k,t) \propto D_1^2(t) \propto a^2$, we find

$$P(k,t) = P(k,t_c)\frac{a^2(t)}{a^2(t_c)} = \frac{\delta_H^2}{4\pi k^3}\frac{a^2(t)}{a^2(t_c)} \propto \delta_H^2 a^2 k.$$

Bottom line: scale-invariant fluctuations produce $P(k,t) \propto k$ on scales that reenter the Hubble radius during the matter dominated era.

*The transfer function*

But comoving scales smaller than $\lambda_{\rm eq} \sim ct_{\rm eq}(1+z_{\rm eq})$, where eq denotes the epoch of equal matter and radiation densities, reenter the Hubble radius when the universe is radiation dominated.

These fluctuations grow slower than $a(t)$ until the universe becomes matter dominated.

For very small scales, which reenter when the universe is strongly radiation dominated, we get essentially no growth after reentering the Hubble radius.

For wavelengths in this regime, at *fixed time*

$$4\pi k^3 P(k,t) \approx 4\pi k^3 P(k,t_c) \approx \text{const.} \implies P(k,t) \propto k^{-3}$$

(in practice, there is logarithmic growth, hence an additional factor of $\log k$.)

The transition between scales that enter in the radiation dominated regime and scales that enter in the matter dominated regime is a slow one, so the change from $P(k) \propto k$ to $P(k) \propto k^{-3}$ is gradual.

The linear theory CDM power spectrum is written

$$P(k,t) \propto k T^2(k) D_1^2(t),$$

where $T(k)$ is the *transfer function.*

$T(k)$ describes the growth of fluctuations after they re-enter the horizon. It depends on the energy density of components with different equations of state, ($\Omega_{\rm CDM} h^2$, $\Omega_b h^2$, $\Omega_\Lambda h^2$, $\Omega_r h^2$, $\Omega_\nu h^2$, ...), but it applies regardless of what spectrum of fluctuations comes out of inflation (or any other mechanism that produces primeval fluctuations).

$T(k)$ does depend on whether fluctuations at horizon crossing are adiabatic (matter and radiation fluctuate together) or isocurvature (opposite matter and radiation fluctuations), since this determines the interplay between gravity and pressure.

From our above analysis, we see that $T^2(k) = $const. at large scales and $T^2(k) \propto k^{-4}$ at small scales. The transition where $P(k)$ turns over is (in *comoving* units)

$$\lambda \sim c t_{\rm eq}(1+z_{\rm eq}) = 50 \left(\frac{\Omega_m h^2}{0.15}\right)^{-1} \text{Mpc} = 36 \left(\frac{\Omega_m h}{0.21}\right)^{-1} h^{-1}\,\text{Mpc}.$$

In practice, this scale is slightly modified by the gravitational effects of the baryon component.

*Tilt, running, broken scale invariance*

Since inflation may not produce perfectly scale-invariant fluctuations, the spectrum is often written

$$P(k,t) \propto k^n T^2(k) D_1^2(t),$$

where $n = 1$ corresponds to scale-invariant inflationary fluctuations and $n \neq 1$ is called a "tilted" spectrum.

Typical inflation models predict $n = 0.9 - 1$, but larger tilts (and $n > 1$) are possible.

The effective value of $n$ could also change with scale; this is called "running of the spectral index." It is expected to be small in typical inflation models.

A feature in the inflaton potential can also produce "broken" scale invariance, with a sharp feature in the power spectrum.

*Hot and warm dark matter*

If the dark matter is mildly relativistic at the time that a scale reenters the Hubble radius, then it will stream preferentially out of overdense regions and into underdense regions, ironing out fluctuations on that scale.

If the dark matter consisted of neutrinos with rest mass $\sim 10 - 30$ eV, then they would be mildly relativistic at $t_{\rm eq}$ (essentially by definition, since their number densities are similar to those of photons), and fluctuations on scales smaller than $ct_{\rm eq}$ would be strongly suppressed, radically changing $T(k)$.

This gives a "hot dark matter" power spectrum. A pure hot dark matter model fails because the absence of small scale fluctuations means that small objects are unable to form at high redshift.

If the dark matter particle has a mass $\sim 1$ keV, then primordial fluctuations are suppressed on sub-galactic scales but not on galactic scales.

Galaxies can still form fairly early, but they are not preceded by generations of smaller collapsed structures.

This "warm" dark matter model is still viable and offers some attractive features.

However, the "cold" dark matter model, in which particles are massive enough that fluctuations are not erased on any cosmologically interesting scale, seems more natural, because the particle mass does not have to be fine-tuned and lies more in the range expected from particle physics models.

*Baryon wiggles*

Before recombination, baryons are tightly coupled to photons. The baryon-photon fluid has very high pressure (sound speed $\sim c/\sqrt{3}$).

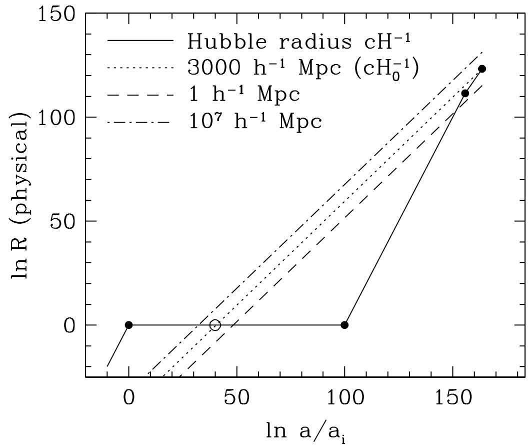
This pressure modifies fluctuation evolution: fluctuations oscillate as sound waves instead of growing steadily.

If there were no dark matter, these oscillations would produce large wiggles in $T(k)$, and diffusion of photons would suppress fluctuations on small scales.

This pure baryonic model is inconsistent with observations.

However, there should still be low amplitude oscillations in $P(k)$ because the oscillating baryons tug on the dark matter.

The strength of the oscillations depends on the ratio of baryons to dark matter.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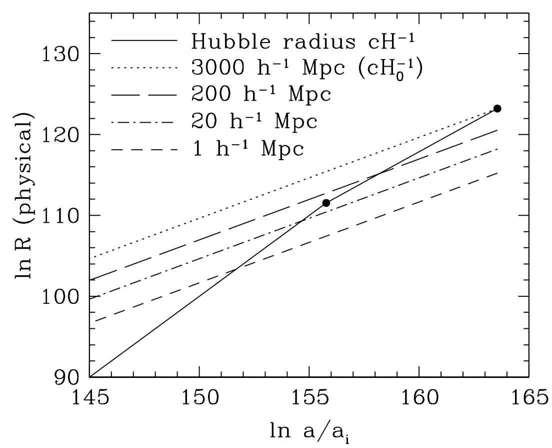
3000 h⁻¹ Mpc (cH₀⁻¹)
1 h⁻¹ Mpc
10⁷ h⁻¹ Mpc
ln R (physical)
ln a/a_i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200 h⁻¹ Mpc
20 h⁻¹ Mpc
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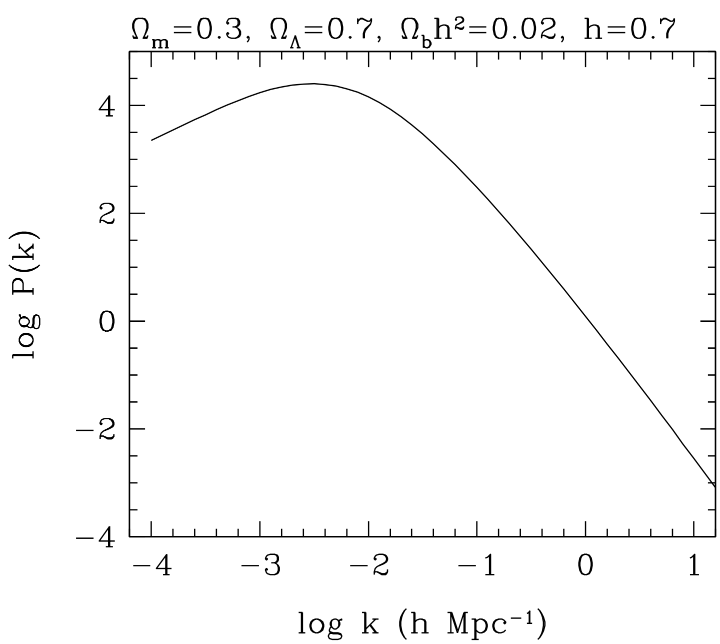
ln R (physical)
ln a/a_i


Ωm=0.3, ΩΛ=0.7, Ωbh²=0.02, h=0.7
log P(k)
log k (h Mpc⁻¹)