Problem Set 2: Cosmological Parameters and Their Evolution
Due Wednesday, October 24

Recall that

$$\text{the redshift} \quad z \equiv \frac{a_0}{a} - 1$$
$$\text{the Hubble parameter} \quad H \equiv \frac{\dot{a}}{a}$$
$$\text{the critical density} \quad \rho_c \equiv \frac{3H^2}{8\pi G} \, ,$$
$$\text{the density parameter} \quad \Omega \equiv \frac{\rho}{\rho_c} \, .$$

where a_0 is the value of the expansion factor at $z = 0$. In our standard notation (adopted below), $a_0 \equiv 1$, so $z = a^{-1} - 1$.

The Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2 R_0^2}, \qquad k = -1,\ 0,\ \text{or}\ +1,$$

where ρ is the *total* energy density.

(1) Write the Friedmann equation in terms of H and Ω (instead of $\dot{a}$ and ρ). What is the relation between the values of Ω and k? For $k \neq 0$, what is the value of the curvature radius $R = aR_0$ in terms of H and Ω?

(2) Now assume that the energy components of the universe are matter, radiation, and a constant vacuum energy (i.e., a cosmological constant). The energy densities of these components scale with redshift as

$$\rho_m = \rho_{m,0}(1+z)^3$$
$$\rho_r = \rho_{r,0}(1+z)^4$$
$$\rho_\Lambda = \rho_{\Lambda,0} = \text{const.},$$

where subscripts zero denote values at $z = 0$. Define

$$\Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_r = \frac{\rho_r}{\rho_c}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}$$

implying

$$\Omega = \Omega_m + \Omega_r + \Omega_\Lambda.$$

Further define

$$\Omega_k = -\frac{kc^2}{a^2 R_0^2 H^2}.$$

[Note that $\Omega_k = -kD_H^2/(a^2R_0^2)$ where $D_H(z) \equiv cH^{-1}(z)$ is the Hubble distance.]

Use the Friedmann equation to show that

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1.$$

Caution: In the older cosmological literature, Ω without subscript is often used to denote Ω_m alone. The convention adopted here follows the now mostly standard but not quite universal trend to have Ω refer to the contribution of all energy components and denote the matter density parameter Ω_m.

(3) Show that the Friedmann equation can be written in the form

$$H(z) = H_0 \left[\Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4\right]^{1/2}, \tag{1}$$

where subscripts 0 refer to the present day values of parameters. (Hint: Draw on your solution from part (1) and think carefully about how ρ and ρ_c change with redshift.)

(4) Show that the lookback time to an object at redshift z (i.e., the elapsed time from redshift z to redshift 0) is

$$t = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')\left[\Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4\right]^{1/2}}. \tag{2}$$

(5) Show that

$$\Omega_m(z) = \frac{\Omega_{m,0}(1+z)^3}{\Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4}. \tag{3}$$

(6) Assume that $\Omega_{r,0}$ and $\Omega_{\Lambda,0}$ are negligibly small (i.e., they can be set to zero in equation 3). What are the asymptotic values of Ω_m and Ω_k for $z \gg \Omega_{m,0}^{-1}$?

Plot $\Omega_m(z)$ in this case for $k = +1,\ 0,\ -1$. I prefer a schematic *hand drawn* plot to a computer-generated plot of the equation, but be sure to label axes, mark relevant values, and show all of the relevant features of the behavior.

(7) Assume instead that $\Omega_{r,0}$ and $\Omega_{k,0}$ are negligibly small, but $\Omega_{\Lambda,0}$ is not. What are the asymptotic values of Ω_m and Ω_Λ for $z \gg \Omega_{m,0}^{-1}$?

Make a schematic plot of $\Omega_m(z)$ comparing the case $\Omega_{m,0} = 0.3$, $\Omega_{k,0} = 0$, $\Omega_{\Lambda,0} = 0.7$ to the case $\Omega_{m,0} = 0.3$, $\Omega_{k,0} = 0.7$, $\Omega_{\Lambda,0} = 0$.

(8) The space density of bright galaxies (as bright as or brighter than the Milky Way) is about 0.01 galaxies/Mpc^3. Several of the best current measurements of the Hubble constant yield $H_0 \approx 70$ km/s/Mpc. *Assume* that (a) the above value for H_0 is correct, (b) the average mass of a bright galaxy is $2 \times 10^{12} M_\odot$ (including the dark matter halo), and (c) essentially all of the matter in the universe is attached to bright galaxies. What is the value of $\Omega_{m,0}$?
[1 Mpc = 3.09×10^{24} cm. 1 $M_\odot$ = 1 solar mass = 2×10^{33} g.]

(9) With the same assumptions, what is the ratio of the matter energy density $\rho_{m,0}$ to the energy density $\rho_{r,0}$ of the 2.73° radiation background? At what redshift were ρ_m and ρ_r equal?

(10) The "Planck density" is $\rho_{\rm Planck} = c^5/(\hbar G^2)$, the only combination of the fundamental constants c, G, and $\hbar$ that has units of density. What is the Planck density in g cm^{-3}? If $\rho_{\Lambda,0}$ is equal to the Planck density and $H_0 = 70$ km/s/Mpc, what is $\Omega_{\Lambda,0}$?