## III. Homogeneous Cosmological Models: Geometry

### Readings

The material in this section and the next is covered in chapters 2 and 3 of Huterer. It is also covered in chapters 3-7 of Ryden's textook, which I think is particularly good on this subject.

I very much like the discussion in Jim Gunn's (1978) article *The Friedmann Models and Optical Observations in Cosmology*, from the SAAS-FEE Proceedings *Observational Cosmology*, edited by A. Maeder et al. You can find this through NASA ADS. This article is particularly good on showing how far you can get with relativity short of the Friedmann equation.

For distance measures, I recommend the compact summary by David Hogg, *Distance Measures in Cosmology*, available as astro-ph/9905116. It generally does not give derivations, but it summarizes the standard results in admirably clear form.

Notation differs from one treatment to another. Mine is similar to Ryden's, which I consider superior to Huterer's.

There are lots of equations in this section. I have tried to number the ones that are important end results as opposed to steps along the way.

### The Einstein cosmological model

*Cosmological Considerations on the General Theory of Relativity* (Einstein, 1917, reprinted in the *Principle of Relativity* book)

Einstein constructs the first modern cosmological model, drawing on new concepts of relativity.

Gives arguments, not particularly persuasive, against an infinite universe.

Introduces hypothesis: universe is homogeneous on large scales.

How can universe be finite and homogeneous? What about boundary?

GR allows a solution: space is positively curved, like the surface of a sphere. Finite, but no boundary.

Seeks static solution with constant $\rho$, positive curvature.

Proves that no such solution exists.

In response, Einstein *modifies* the field equation from

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \qquad \text{to} \qquad G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu},$$

adding the "cosmological term."

For a specific relation between the cosmological constant $\Lambda$ and the matter density $\rho$, this allows a static solution.

The cosmological models subsequently developed by Friedmann and LeMaître retain some ideas from the Einstein cosmology — GR, large scale homogeneity, large scale curvature — but they drop the assumption that the universe is static. They therefore do not *require* a cosmological constant.

Einstein abandoned the cosmological term for good when the cosmic expansion was discovered (in 1929). He is reputed to have called it "the greatest blunder of my life."

### The Cosmological Constant

The cosmological constant idea has never completely gone away. It has been especially prominent in the last 20 years, but it is now viewed as part of $T_{\mu\nu}$.

Instead of

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \qquad G_{\mu\nu} = 8\pi(T_{\mu\nu}^{\rm matter} + T_{\mu\nu}^{\rm VAC}), \quad \text{where}$$
$$T_{\mu\nu}^{\rm VAC} \equiv (\Lambda/8\pi) g_{\mu\nu}.$$

Recall that the ideal fluid $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$.

$T_{\mu\nu}^{\rm VAC}$ is the stress-energy tensor of a "false vacuum" or "scalar field" with equation of state $p = -\rho = \Lambda/8\pi$.

The basic effect can be seen from our Newtonian limit result:

$$\nabla^2\Phi = 4\pi(\rho + 3p).$$

For $p < -\rho/3$, gravity pushes instead of pulls.

With the right choice of $\Lambda$, can have static model in which push of vacuum energy balances pull of matter. However, this balance is unstable to small changes in the matter or vacuum energy density.

With larger $\Lambda$, get acceleration.

### The Friedmann-Robertson-Walker metric

The *spatial* metric of the Einstein cosmological model is that of a 3-sphere:

$$dl^2 = dr^2 + R^2 \sin^2(r/R) d\gamma^2$$

where $d\gamma^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ is the angular separation.

Here $R$ is the curvature radius of the 3-dimensional space, and $r$ is distance from the origin.

In the coordinate frame of a freely falling observer, time is just proper time as measured by the observer, and the spacetime metric is

$$ds^2 = -c^2 dt^2 + dl^2.$$

A natural generalization of the Einstein model is to allow the curvature radius $R(t)$ to be a function of time.

The universe is still homogeneous and isotropic on a surface of constant $t$, but it is no longer static.

In the 1930s, Robertson and Walker (independently) showed that there are only three possible spacetime metrics for a universe that is homogeneous and isotropic.

They can be written

$$ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + S_k^2(r)d\gamma^2\right], \tag{3.1}$$

where

$$S_k(r) = R_0 \sin(r/R_0), \qquad k = +1, \tag{3.2}$$
$$= r, \qquad k = 0, \tag{3.3}$$
$$= R_0 \sinh(r/R_0), \qquad k = -1. \tag{3.4}$$

In this notation, $a(t)$ is dimensionless.

It is defined so that $a(t_0) = 1$ at the time $t_0$ (usually taken to be the present) when the curvature radius is $R_0$. At other times the curvature radius is $a(t)R_0$.

The radial coordinate $r$ and the radius of curvature $R_0$ have units of length (e.g., Mpc).

I have followed Ryden's notation in giving $S_k(r)$ units of length. In Gunn's notation, $S_k(r)$ is dimensionless, and $a(t)$ is replaced by $R(\tau)$, where $R(\tau)$ has units of length.

For $k = +1$, the space geometry at constant time is that of a 3-sphere.

For $k = 0$, the space geometry at constant time is Euclidean, a.k.a. "flat space."

For $k = -1$, the space geometry at constant time is that of a negatively curved, 3-dimensional "pseudo-sphere."

Friedmann and LeMaître used this metric in their cosmological models of the 1920s. Robertson and Walker proved that they are the only forms consistent with the Cosmological Principle (homogeneity and isotropy).

It is commonly called the Friedmann-Robertson-Walker metric, or sometimes the Robertson-Walker metric.

**The FRW metric, space curvature, and spacetime curvature**

$k = +1 \Longrightarrow$positive curvature, spherical geometry, finite space

$k = 0 \Longrightarrow$no curvature, Euclidean geometry, infinite space

$k = -1 \Longrightarrow$negative curvature, pseudo-sphere geometry, infinite space

Note: these are descriptions of *space* at constant $t$.

For many forms of $a(t)$, *spacetime* is positively curved even if *space* is not, (this is always the case unless a cosmological constant or some other form of energy with negative pressure is important). In the special relativistic, Milne cosmology, spacetime is flat, but surfaces of constant time are negatively curved.

Positive curvature $\Longrightarrow$geodesics "accelerate" (in 2nd derivative sense) towards each other. Initially "parallel" geodesics converge.

Example: great circles on a sphere.

Zero curvature $\Longrightarrow$no geodesic "acceleration." Initially parallel geodesics stay parallel. Euclidean geometry.

Example: straight lines on a plane.

Negative curvature $\Longrightarrow$geodesics "accelerate" away from each other. Initially parallel geodesics diverge.

Example: geodesics on a saddle.

The substitution $x = S_k(r)$ allows the FRW metric to be written in another frequently used form:

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dx^2}{1 - kx^2/R_0^2} + x^2d\gamma^2\right].$$

Demonstration is left as a (simple) exercise for the reader.

With $r$ as radial coordinate, radial distances are "Euclidean" but angular distances are not (unless $k = 0$). With $x$ as radial coordinate, the reverse is true.

**Comoving Observers**

The metric depends on the coordinate frame of the observer.

Even a homogeneous and isotropic universe only appears so to a special set of freely falling observers, called "Fundamental Observers (FOs)" or "Comoving Observers."

These observers are "going with the flow" of the expanding universe, and the proper distance between them increases in proportion to $a(t)$.

In the coordinate frame of these observers, the FRW metric applies, and the time coordinate of the FRW metric is just proper time as measured by these observers.

Comoving spatial coordinates track the positions of these FOs, i.e., the comoving separation between any pair of FOs remains constant in time.

An observer moving relative to the local FOs has a "peculiar velocity," where peculiar is used in the sense of "specific to itself" rather than "odd."

An observer with a non-zero peculiar velocity does not see an isotropic universe – e.g., dipole anisotropy of the cosmic microwave background caused by reflex of the peculiar velocity.

**Examples of application of metric**

(1) Any freely falling particle follows geodesics in spacetime, whose solution in comoving coordinates could be found from the geodesic equation. For comoving particles (FO's), solution is trivial: $r,\ \theta,\ \phi =$ constant.

(2) Light rays travel along null geodesics: $ds^2 = 0$. Therefore, along a radial ray ($d\gamma = 0$), $dr = c\,dt/a(t) \implies r_o - r_e = \int_{t_e}^{t_o} c\,dt/a(t)$.

(3) In a surface of constant $t$, metric distance along a radial path of constant $\theta, \phi$ is

$$l = \int ds = \int_{r_1}^{r_2} a(t)dr = a(t)(r_2 - r_1).$$

(4) In a surface of constant $t$, metric distance along a path of constant $r, \theta$ between two points of different $\phi$ is

$$l = \int ds = \int_{\phi_1}^{\phi_2} a(t)S_k(r)d\gamma = a(t)S_k(r)\sin\theta(\phi_2 - \phi_1).$$

Note that this is not a great circle (and hence shortest) path unless $\theta = \pi/2$.

(5) In a surface of constant $t$, metric volume of a shell of radius $r$ and width $\Delta r \ll r$ is

$$V = \int d^3s = \int_r^{r+\Delta r} a(t)dr \int_{4\pi} a^2(t)S_k^2(r)d\gamma^2 = 4\pi S_k^2(r)a^3(t)\Delta r.$$

For $k = 0$ this is just $4\pi r^2 \Delta r \times a^3(t)$.

Note that the "metric" distances and volumes in (3)-(5) are "proper," physical distance measures and that $r$ is the *comoving* radial coordinate.

**Redshift of photons**

The physical distance to a galaxy at comoving distance $r$ is $d = a(t)r$.

The Hubble parameter is

$$H = \dot{d}/d = \dot{(ar)}/(ar) = \dot{a}/a. \tag{3.5}$$

A photon emitted by a nearby comoving source at distance $d$ is Doppler shifted:

$$\frac{d\nu}{\nu} = \frac{-v}{c} = \frac{-Hd}{c} = -Hdt,$$

where the last equality follows because $d = c\,dt$.

Thus,

$$\frac{d\nu}{\nu} = -\frac{\dot{a}\,dt}{a} = -\frac{da}{a} \implies d\ln\nu = -d\ln a.$$

Let the photon be emitted with frequency $\nu_e$ at time $t_e$ and observed with frequency $\nu_o$ at time $t_o$.

Integrate to get

$$\frac{\nu_e}{\nu_o} = \frac{a_o}{a_e} = \frac{\lambda_o}{\lambda_e} \equiv (1+z), \qquad z = \text{redshift}. \tag{3.6}$$

Constant of integration fixed by demanding $\nu_o \longrightarrow \nu_e$ as $a_o \longrightarrow a_e$.

Photon wavelength proportional to $a(t)$.

Frequency shift $\Longrightarrow$time dilation. Real effect observed in, e.g., supernova light curves.

Could also derive by (a) considering successive crests traveling on null geodesics, or (b) using equation for evolution of 4-momentum along null geodesic.

As with gravitational redshift, cosmological redshift implies time dilation. This is seen in, e.g., the light curve of high-redshift supernovae. In the frame of the supernova, the characteristic rise and fall time is (presumably) the same as for nearby supernovae, but the observed rise and fall of high-redshift supernovae takes longer.

**Kinematic redshift**

Give a particle a "peculiar" velocity with respect to the comoving frame.

The peculiar velocity decays as it catches up with receding particles.

This is a purely kinematic effect, though it looks like a fictitious "friction."

In the non-relativistic limit, a particle with speed $v$ goes distance $v\,dt$ and changes its peculiar velocity by $dv = -H(v\,dt)$

$$\Longrightarrow \quad \frac{dv}{v} = -\frac{\dot{a}\,dt}{a} = -\frac{da}{a} \quad \Longrightarrow \quad \frac{p_e}{p_o} = \frac{mv_e}{mv_o} = \frac{a_o}{a_e}.$$

Momentum redshifts like the frequency of a photon.

Gunn shows that this continues to hold in the relativistic case ($dv$ large).

One can also show that this implies that the de Broglie wavelength of a particle redshifts just like photon wavelengths.

Kinematic redshift profoundly affects the dynamics of instabilities: in an expanding universe (or any expanding medium), undriven disturbances decay instead of coast.

**Flux, diameter, and surface brightness vs. redshift**

$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + S_k^2(r)d\gamma^2].$$

From metric application (2) above, we have the comoving distance to an object that emitted light at time $t_e$ as

$$D_c = r = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)}.$$

From $a \equiv (1+z)^{-1}$ we have $da = -dz(1+z)^{-2} = -a^2 dz$, and from $H \equiv \dot{a}/a$ we have $dt = da/(aH) \Longrightarrow dt/a(t) = da/(a^2 H) = -dz/H$.

Putting these results together yields

$$D_c = \int_0^z \frac{c\,dz'}{H(z')} \;=\; \frac{c}{H_0}\int_0^z dz' \frac{H_0}{H(z')} \tag{3.7}$$

for the comoving distance to an object at redshift $z$.

The characteristic distance $c/H_0$ is sometimes referred to as the Hubble distance

$$D_H \equiv \frac{c}{H_0} = 13.95 \left(\frac{70\text{km s}^{-1}\text{ Mpc}^{-1}}{H_0}\right) \text{Glyr} = 4.28 \left(\frac{70\text{km s}^{-1}\text{ Mpc}^{-1}}{H_0}\right) \text{Gpc} \tag{3.8}$$

Using the relation for $H_0/H(z')$ that we will derive later from the Friedmann equation then reproduces equation (15) of Hogg (1999).

An object of angular size $d\gamma$ at time $t_e$ has a transverse physical size

$$dl = a(t_e)S_k(r)d\gamma = (1+z)^{-1}S_k(r)d\gamma. \tag{3.9}$$

The angular diameter distance is

$$D_A = \frac{dl}{d\gamma} = (1+z)^{-1}S_k(r),$$

where $r$ is the comoving distance $D_c$ as given above.

We will later find from the Friedmann equation that the curvature radius is

$$R_0 = \frac{c}{H_0}|\Omega_k|^{-1/2}, \tag{3.10}$$

where $\Omega_{\rm tot} = 1 - \Omega_k$ is the ratio of the total energy density of the universe (mass, radiation, dark energy, ...) to the critical density. For $\Omega_k \to 0$, $R_0 \to \infty$ and the curvature $1/R_0^2 \to 0$.

Together with the definition $D_H = c/H_0 \Longrightarrow R_0 = D_H|\Omega_k|^{-1/2}$, this yields the result of equations (16) and (18) of Hogg (1999):

$$\begin{aligned} D_A &= D_H(1+z)^{-1}\Omega_k^{-1/2}\sinh(\Omega_k^{1/2}D_c/D_H) \qquad && k=-1, \\ &= D_c(1+z)^{-1} && k=0, \\ &= D_H(1+z)^{-1}|\Omega_k|^{-1/2}\sin(|\Omega_k|^{1/2}D_c/D_H) && k=+1. \end{aligned}$$

The photons from a source at redshift $z$ are distributed over an area $4\pi S_k^2(r)$ at the present day, since $a(t_0) \equiv 1$.

Photons emitted in a time $dt_e$ are received over an interval $dt_0 = (1+z)dt_e$, and they are shifted downward in energy by $(1+z)$. The bolometric flux $F$ is therefore reduced by an additional factor $(1+z)^2$:

$$F = \frac{L}{4\pi S_k^2(r)(1+z)^2} \equiv \frac{L}{4\pi D_L^2}, \tag{3.11}$$

where $L$ is the source bolometric luminosity and $D_L = (1+z)^2 D_A$ (Hogg 1999, eq. 21).

In c.g.s. units,

$$[F] = \text{erg s}^{-1}\ \text{cm}^{-2}, \qquad [L] = \text{erg s}^{-1}.$$

The solid angle subtended by a source of projected area $A$ is $\Omega = A/D_A^2$, making the surface brightness

$$I_0 \equiv \frac{F}{\Omega} = \frac{L}{4\pi D_L^2}\frac{D_A^2}{A} = \frac{L}{4\pi A}\frac{1}{(1+z)^4} = \frac{I_e}{(1+z)^4}.$$

This is the famous $(1+z)^4$ surface brightness dimming of cosmological sources, which can make high redshift galaxies very difficult to detect.

Note, however, that these relations for flux and surface brightness are bolometric, integrated over all wavelengths.

The relation for monochromatic fluxes can be written

$$\nu_0 S_{\nu_0} = \frac{\nu_e L_{\nu_e}}{4\pi D_L^2} = \frac{\nu_e L_{\nu_e}}{4\pi S_k^2(1+z)^2}. \tag{3.12}$$

The monochromatic or passband flux of an astronomical object is affected by the redshifting of the bandpass from $\nu_0$ to $\nu_e = \nu_0(1+z)$.

This effect is referred to as the $K$-correction (see Hogg et al. 2002, astro-ph/0210394).

**Most important fact about redshift:**

*If we measure the redshift of a source (e.g., from the frequency of a spectral line), we know* $a_o/a_e$.

Given a model of $a(t)$, we also know the radial distance $D_c$. For angular diameter and luminosity distances, we also need to know the space curvature ($R_0$ and $k$, or $\Omega_k$).

Nearby (i.e., to first order in $z$),

$$cz = c\frac{\lambda_o - \lambda_e}{\lambda_e} = H_0 d, \qquad H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)},$$

independent of other cosmological parameters.

Alternatively, if we can infer distance from observed flux (of a source of known luminosity) or observed angular size (of a source of known physical size), we can reconstruct $a(t)$ from observations, and constrain cosmological parameters.

Note: If we see an object at redshift $z$, or a separation of two objects at redshift $z$, we sometimes refer to the physical or "proper" size/separation and sometimes to the comoving size separation. The comoving size/separation is larger by $(1+z)$. Papers sometimes clarify this by specifying units as, e.g., pkpc or cMpc.