Problem Set 4: WIMP Dark Matter Freeze-out
Due Wednesday, March 20

Introduction

The goal of this problem set is to compute the relic abundance of thermal WIMPs (weakly interacting massive particles) from the early universe. I've formulated this in an approximate way that I think is basically correct and gives a reasonable answer, but it's not a form I've seen elsewhere so there could be errors. The definitive calculation is the highly cited paper by Steigman, Dasgupta, and Beacom (2012), which is a more sophisticated version of the calculation described in §6.1.2 and Box 11.2 of Huterer.

Unless otherwise specified, quantities like m and n refer to the mass and number densities of WIMPs.

For early universe problems, it is often convenient to adopt "high energy physics" units in which $\hbar = c = k_B = 1$ ($k_B =$ Boltzmann's constant) and the fundamental dimension is energy. One goal of this problem set is to give you experience in doing a calculation this way.

A traditional and convenient unit of energy is 1 GeV$=$ 10^9 eV, and in the high energy system of units:

$$1 \text{ GeV} \;=\; 1.16 \times 10^{13} \text{ K} \;=\; 1.78 \times 10^{-24} \text{ g} \;=\; (1.97 \times 10^{-14} \text{ cm})^{-1} \;=\; (6.58 \times 10^{-25} \text{ s})^{-1}. \tag{1}$$

Newton's gravitational constant enters into calculations via the Planck mass,

$$m_{\rm Pl} \equiv (\hbar c/G)^{1/2} = G^{-1/2} = 2.18 \times 10^{-5} \text{ g} = 1.22 \times 10^{19} \text{ GeV}. \tag{2}$$

As discussed in class and in Huterer §4.2, the energy density of relativistic particles can be written

$$\rho = \frac{\pi^2}{30} g_* T^4, \tag{3}$$

where

$$g_* \equiv \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T_\gamma}\right)^4 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4, \tag{4}$$

the sums are over all species of particles relativistic at temperature T_i, and we have allowed for the possibility that each species i is characterized by a different temperature T_i. Figure 4.1 of Huterer shows the evolution of g_* as a function of time t and temperature T.

WIMPs are non-relativistic at the time of freeze-out, so their number density is (Huterer 4.23)

$$n = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \;. \tag{5}$$

1. Friedmann equation

Using the Friedmann equation for a $k = 0$, radiation dominated universe, show that the Hubble parameter $H(T)$ at the time that the temperature of relativistic species is T is

$$H = \left(\frac{\dot{a}}{a}\right) = 1.66 g_*^{1/2} \frac{T_\gamma^2}{m_{Pl}}. \tag{6}$$

2. Freeze-out condition

Approximate WIMP freeze-out by assuming that WIMP/anti-WIMP annihilation maintains the number density at the thermal equilibrium value of eq. (5) until the annihilation rate falls to

$$\Gamma_f = n_f \langle \sigma v \rangle = \alpha_f H \ , \tag{7}$$

after which WIMP annihilation stops completely and the number density changes only by dilution from cosmic expansion. Here the subscript f denotes freeze-out, n_f is the WIMP number density at freeze-out, $\langle \sigma v \rangle$ is the thermally averaged annihilation cross-section, and α_f is a dimensionless fudge factor we are carrying to check sensitivity to the details of this assumption.

If the freeze-out temperature is high enough that many particle species are relativistic, we have $g_*^{1/2} \approx 10$ ($g_* \approx 100$, see Huterer Fig. 4.1). Adopting this value for $g_*^{1/2}$ and $g = 2$ for WIMPs, show that the freeze-out condition (7) implies

$$\left(\frac{m}{T_f}\right)^{3/2} e^{-m/T_f} = \frac{5 \times 1.66 \times (2\pi)^{3/2} \alpha_f}{\langle \sigma v \rangle m_{\rm Pl} T_f} \ . \tag{8}$$

3. Estimate m/T_f

For the thermally averaged cross-section, we will take a value typical (at order-of-magnitude level) for weak interactions,

$$\langle \sigma v \rangle = 3 \times 10^{-9} \mathrm{GeV}^{-2} \ .$$

Note that the combination $\langle \sigma v \rangle m_{\rm Pl} T_f$ is dimensionless.

By taking the natural-log of both sides of equation (8), write an expression for the ratio m/T_f.

This expression is not one you can solve analytically, but you can easily find an approximate answer by trial and error. Taking $T_f = 4\mathrm{GeV}$ as a guess, and assuming $\alpha_f = 1$, use your expression to show that

$$\frac{m}{T_f} \approx 30 \ .$$

How sensitive is your result to the assumed values of T_f and α_f? Why is this sensitivity so weak?

Optional: Compare this value to Fig. 1 of Steigman et al. (2012).

4. Number density at freeze-out

Go back to the freeze-out condition (7). Show that the number density of WIMPs at freeze-out is

$$n_f = \frac{1.66 g_*^{1/2} \alpha_f T_f^2}{m_{\rm Pl} \langle \sigma v \rangle} \ . \tag{9}$$

5. Number density today

As discussed in Huterer §4.3 and briefly in class, the quantity

$$g_{*S} T^3 a^3 = \text{const.} \tag{10}$$

throughout cosmic evolution, where g_{*S} is defined like g_* in equation (4) but with 3rd-powers instead of 4th-powers. Based on Huterer Fig. 4.1, take $g_{*S}(T_f)/g_{*S}(T_0) \approx 100/4 = 25$, where $T_0 = 2.39 \times 10^{-4}\,\text{eV}$ is the CMB temperature today.

Show that the current day WIMP density is

$$n_0 \approx n_f \times \frac{1}{25} \left(\frac{T_0}{T_f} \right)^3 \ . \tag{11}$$

6. Mass density today

By combining equation (11) with the n_f formula (9) and the WIMP mass m, show that the present-day mass density of WIMPs is

$$\rho_0 = \frac{1.66 g_*^{1/2} \alpha_f}{25} \times \frac{T_0^3}{m_{\rm Pl} \langle \sigma v \rangle} \times \frac{m}{T_f} \ . \tag{12}$$

What are the units of this expression? Are these correct units for a mass density? (Look back at equation 1.)

7. Ratio to the radiation density

We could restore the necessary constants, evaluate equation (12) in $\text{g}\,\text{cm}^{-3}$, and compare to the critical density. Instead we'll take a ratio to the present-day energy density of the CMB, $\rho_{\gamma,0}$, and use the empirical result (see Huterer Table 3.1) that

$$\Omega_\gamma h^2 = 2.47 \times 10^{-5} \ . \tag{13}$$

Recall that

$$\rho_{\gamma,0} = \frac{\pi^2}{15} T_0^4$$

(using $g = 2$ for photons). Demonstrate that

$$\frac{\rho_0}{\rho_{\gamma,0}} = \frac{1.66 g_*^{1/2} 15 \alpha_f}{25\pi^2} \times \frac{1}{m_{\rm Pl} T_0 \langle \sigma v \rangle} \times \frac{m}{T_f} \ . \tag{14}$$

8. The dark matter density today

Evaluating equations (13) and (14), what is the predicted value of the WIMP density $\Omega_0 h^2$?

How does the predicted $\Omega_0 h^2$ depend on the assumed WIMP mass? (Think carefully.)

How does the predicted $\Omega_0 h^2$ depend on the assumed α_f?

How does the predicted $\Omega_0 h^2$ depend on the assumed $\langle \sigma v \rangle$? Explain the direction of this dependence.

What other comments do you have on this result or your derivation of it?