LECTURE 6 (1/17): PRINCIPLES OF STELLAR STRUCTURE
I. WHY DON'T STARS COLLAPSE?

## 6.1 WHY DOESN'T THE SUN COLLAPSE?

How long would a satellite just above the surface of the sun take to go around it?
The solar radius is about 0.005 AU. Applying Kepler's law, $P^2 = a^3$, we find that the period at 0.005 AU is 0.00035 years, about 3 hours.

If nothing supported the sun against its own gravity, it would collapse in a few hours. Why doesn't this happen?

Answer: pressure balances gravity.

## IMPORTANT EQUATIONS

### # 4: THE EQUATION OF STATE FOR AN IDEAL GAS

In symbols:

$$P = knT.$$

$P$ = pressure of the gas [dynes/cm$^2$]
$n$ = number density [atoms/cm$^3$]
$T$ = temperature [degrees Kelvin]
$k$ = Boltzmann's constant

In words: The pressure exerted by an ideal gas is proportional to its density and to its temperature in degrees Kelvin. The constant of proportionality is Boltzmann's constant $k$, which tells how much energy a typical atom has when the temperature is $T$.

Examples:

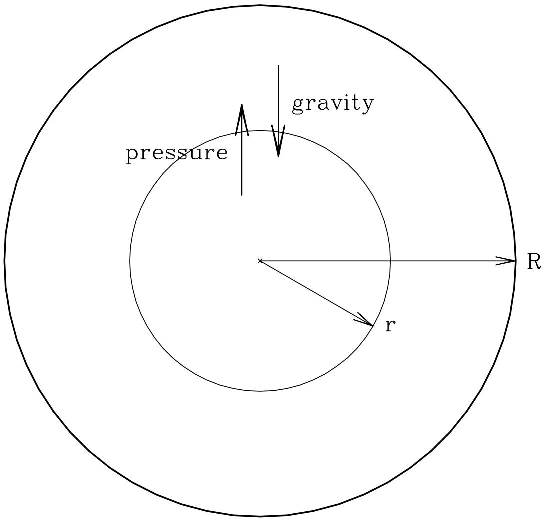
(1) I have some high-pressure oxygen stored in a tank. I siphon half of the oxygen into another tank. If the temperature does not change when I do this, what happens to the gas pressure?
The volume of the tank stays the same, but I reduce the amount of gas by a factor of two, so the density $n$ goes down by a factor of two. Since $T$ doesn't change, the pressure drops by a factor of two.

(2) I have some helium gas in a jar. I heat this jar over a burner so that the temperature of the helium (in degrees Kelvin) doubles. What happens to the pressure?
The density doesn't change, the temperature doubles, so the pressure doubles.

### 6.2 HYDROSTATIC EQUILIBRIUM

The most important principle that governs the structure of stars is *hydrostatic equilibrium.* Translation: pressure balances gravity.

At *every* position in a star, the pressure of the gas must be *just enough* to support the "weight" of the star above it. Otherwise the star would expand or contract on a timescale of hours. (See Figure 12-16).




How does pressure adjust to be exactly what's needed?
If pressure were too low, the star would contract, and the pressure would go up.
If pressure were too high, the star would expand, and the pressure would go down.

### 6.3 THE CENTRAL TEMPERATURE OF THE SUN

Hydrostatic equilibrium – balance of pressure and gravity – must hold throughout the star.

We can apply this principle in a very approximate way to estimate the temperature in the center of the sun. Leaving aside the equations, the steps in the calculation are as follows:

(1) Divide the sun into an inner half and an outer half. We'll demand that the pressure in the inner half support the weight of the outer half.

(2) Estimate the weight of the outer half from Newton's law of gravity.

(3) The gas pressure in the inner half depends on the density of atoms and on the temperature. We estimate the density by dividing the number of atoms in the sun by the volume of the sun.

(4) Finally, we find the temperature that is needed to provide the necessary pressure.

Now we'll use equations, but we'll ignore all factors of 2, $4\pi$, and so forth.

Symbols:
$M$ = mass of sun
$R$ = radius of sun
$P_c$ = pressure in center of sun
$m_a$ = mass of a typical atom
$T_c$ = temperature in center of sun (our goal)

The outer and inner shells both have a mass of about $M$, and a radius of about $R$. Newton's law of gravity tells us that the gravitational force on the outer shell (its "weight") is

$$\frac{GM_1M_2}{r^2} \sim \frac{GM^2}{R^2}.$$

A pressure is a force per unit area, so we must multiply by the surface area of the inner shell, roughly $R^2$, to get the total force exerted by the pressure of the inner shell. This force must balance the weight of the outer shell.

$$P_cR^2 \sim \frac{GM^2}{R^2} \qquad \text{or} \qquad P_c \sim \frac{GM^2}{R^4}.$$

Now we want to use the ideal gas equation, $P = knT$. The number of atoms in the sun is equal to the mass of the sun divided by the mass of a typical atom, $M/m_a$. The average *density* of atoms $n$ is equal to the number of atoms divided by the volume of the sun, so

$$P_c = kn_cT_c \sim k\left(\frac{M/m_a}{R^3}\right)T_c \sim \frac{GM^2}{R^4}.$$

Finally we get

$$T_c \sim \frac{GMm_a}{kR}.$$

Putting in the appropriate numbers gives $T_c \sim 22$ million degrees.

A more accurate calculation would give a central temperature of about 15 million degrees.

## IMPORTANT EQUATIONS

### # 5: THE CENTRAL TEMPERATURE OF A STAR

In symbols:

$$T_c = \left(\frac{G}{k}\right)\frac{Mm_a}{R_{\text{avg}}}$$

$T_c$ = central temperature of star [degrees Kelvin]
$M$ = mass of star [grams]
$m_a$ = average mass of atom [grams]
$R_{\text{avg}}$ = "average" radius of material in star [cm]

$G$ = Newton's gravitational constant
$k$ = Boltzmann's constant

In words: The central temperature of a star is proportional to the mass of the star times the mass of typical atom divided by the "average" radius of the star. The constant of proportionality is Newton's gravitational constant $G$ divided by Boltzmann's constant $k$.

Comment: This equation only holds true for stars in which the central pressure is determined by the ideal gas equation.

Examples:

(1) Star A is twice as massive as star B, but it is also twice as big. Which star has the higher central temperature?

Doubling the mass is canceled by doubling the radius; the stars have the same central temperature.

(2) Stars A and B are the same mass and the same radius, but star A is made entirely of hydrogen and star B entirely of helium. Which star has the higher central temperature?

The mass of a helium atom is about four times the mass of a hydrogen atom. If $M$ and $R$ are the same but $m_a$ is four times higher, the central temperature of star B is four times higher than that of star A.

An implication: If a star becomes more centrally concentrated, its central temperature goes up because $R_{\text{avg}}$ goes down, even if the radius of the surface doesn't change.

## 6.4 SUMMARY AND REMARKS

The pressure in a star must support its weight. This principle, pressure balances gravity, is called hydrostatic equilibrium.

The pressure of an ideal gas is $P = knT$.

To prevent the sun from collapsing, the central pressure must be high. From this we infer that the center of the sun is *hot*, about 15 million degrees. More generally, we find $T_c \propto M/R$ for a star of mass $M$ and radius $R$.

The pressure in the sun increases steadily with depth, just as it does in the ocean. The temperature climbs from 6000 degrees at the surface to 15 million degrees in the center.

The central temperature of the sun is not directly observable. To test our ideas, we must build a more complete theory of the sun and compare it to observations. To do this we will need to know how energy moves in the sun, and how it is generated.