Name ______________________

Astronomy H161 – An Introduction to Solar System Astronomy
Winter Quarter 2009 – Prof. Gaudi
Homework #4

Due Monday, February 9 in class

No late homework will be accepted.

Accurate measurements are required for these exercises. All measurements must be made in cm. Measurements which are "starred" (*) must be made to the nearest mm (or 1/10 of a cm).

It's Saturday night. After a long week at your job making blueprints for the new St. Paul's Cathedral, you stumble into The Drinking Gourd, a popular London pub with a "Big Dipper" logo. The loud and heated discussion at the next table is giving you a headache. You are about to tell the three men there to shut up, when you recognize one of them as Christopher Wren, astronomer, Royal Surveyor, master architect of St. Paul's, and your boss. The year is 1684. Wren is telling another man that he is sure that Kepler's 3rd law means that gravitational force must fall inversely as the square of the distance between the Sun and a planet. Wren draws a diagram for the other man (who you now realize must be Wren's successor as Secretary to the Royal Society, Edmund Halley). See Figure 1. "Look at this circle representing the Earth's orbit" he says to Halley. "I've drawn two arrows to represent the Earth's velocity $v_{\oplus}$, at two different times A and B. I find the difference in these two motions by moving the arrows tail-to-tail and completing the triangle. I call the difference $\Delta v_{\oplus}$."

1) Prove that Kepler's 3rd law implies an inverse-square law for gravity.
   a) Measure the angle (in degrees) between A and B. Divide your result by the angle of a full circle (360°). This is the fraction of 1 Earth orbit (1 year) that has elapsed between A and B. How much time is this (in years)?
   b) Measure the length of $v_{\oplus}$ in Wren's triangle (Fig. 1) in cm. Record your answer.

c) "Notice" says Wren "that if I move $\Delta v_\oplus$ back to the Earth's orbit halfway between the velocity measurements, it points exactly toward the Sun." Identify where Wren has drawn $\Delta v_\oplus$ on the Earth's orbit in Figure 1. Measure (*) the length of $\Delta v_\oplus$ accurate to 0.1 cm. Record your answer.

d) Halley asks about the other circle. Wren says that it represents another planet with a larger circular orbit. Measure the radii of this orbit and the Earth's orbit in cm. Record your answers. How many times larger is the radius of the planet orbit than the Earth's orbit? What is the radius of the planet's orbit in AU? You find this by taking the ratio $a_p/a_\oplus$ where a_p is the radius of the planet's orbit and $a_\oplus$ is the radius of the Earth's orbit.

e) Use Kepler's 3rd law to determine the period P of the planet in years:

$$\left(\frac{P}{\mathrm{yr}}\right)^2 = \left(\frac{a}{\mathrm{AU}}\right)^3$$

First take the number $a_p/a_\oplus$ from (1d) and multiply by itself three times: $a_p/a_\oplus \times a_p/a_\oplus \times a_p/a_\oplus$. Then find the period P such that $P \times P$ equals this product. Record your answer.

f) "I see", says Halley, "then during the time the Earth moves from A to B, the planet moves from A′ to B′." Measure the angle between A′ to B′ in degrees. Divide your result by the angle of a full circle (360°). This is the fraction of 1 planet orbit that has elapsed between A′ and B′. Record your answer.

g) Use the length of time between A and B (same as between A′ and B′) in part (1a) and the fraction of the planet period found in (1f) to estimate the period of the planet (in years). Record your answer. Your result should agree with your estimate in (1e). This shows that Wren has constructed his diagrams properly according to Kepler's 3rd Law. You congratulate him on his excellent diagrams. Wren acknowledges your presence, asks you about your progress on the blueprints, but goes back to his conversation.

h) Wren now draws lines representing the velocity v_p of the planet at points A′ and B′. Measure the length of v_p in cm. Record your answer. Use your result from (1b) to find the ratio $v_p/v_\oplus$ of the planet's velocity to the Earth's velocity.

i) Wren sets the two planet velocities tail-to-tail and completes the triangle by forming the velocity change Δv_p. "See", he says to Halley, "the change in velocity is much smaller for the planet than for the Earth. Also, when I move Δv_p back to the planet's orbit, it still points directly at the Sun." Identify where Wren has placed Δv_p on the planet's orbit. Measure (*) the length of the Δv_p to the nearest mm. Record your answer.

j) Use your results from parts (1c) and (1i) to find the ratio $\Delta v_p/\Delta v_\oplus$. Record your answer.

k) "You see now, Halley, that I have proved that the change in velocity is always in the direction of the Sun and that its magnitude varies inversely as the square of the distance from the Sun." Use your results from (1d) and (1j) to demonstrate that the inverse square law

$$\frac{\Delta v_p}{\Delta v_\oplus} = \frac{1}{(a_p / a_\oplus)^2}$$

follows from Wren's diagram constructed according to Kepler's 3rd law.

"Very well", says Halley, "but your proof is valid only for circular orbits. Real planets travel on ellipses. Can you show that the inverse square law for gravity makes planets travel in ellipses (as Kepler says in his 1st law) and that they trace equal areas in equal times (as he says in his 2nd law)?" Wren sighs. "Alas I cannot. In fact I have offered a reward of 20 pounds sterling to anyone who can." "What!" Halley exclaims, "you offer 20 pounds to solve the most difficult problem in the history of natural philosophy? Are you sure you are not being too generous?" You are only too familiar with Wren's concept of generosity from looking at your weekly paychecks, but before you can say anything, the third man at the table announces that he has already solved this problem. It is Robert Hooke, another of your boss's colleagues from the Royal Society. "However" Hooke says, "I have decided not to publish my solution so as to allow others to see how difficult the problem is." Halley is quite irritated. "You must take us for fools," he says to Hooke, "you have no more solved the problem than

Wren's apprentice here has." Halley points to you. The three men have a good laugh at your expense.

2) "Actually" you say "it is quite easy to extend Wren's diagrammatic method to the case of elliptical orbits". You use your drafting skills to draw an ellipse with the Sun at one focus. See Figure 2.
   a) You use Kepler's 2nd law to mark off planet positions A, B, C, D, and E, which are separated by equal time intervals. Measure the angles θ (in degrees) between A and B, between B and C, between C and D, and between D and E. Record your answers in the table. (Note that the first row of this table has been filled in for you. These numbers are correct. If yours do not agree, you are doing something wrong.)
   b) Measure (*) the distance from the Sun to the midpoints of the arcs AB, BC, CD, and DE to the nearest mm. Record your results in the table.
   c) Kepler's 2nd (equal area) law implies that the product θr^2 (or $\theta \times r \times r$) is a constant, where θ is the angle measured in (2a) and r is the distance measured in (2b). Calculate these values using the measurements you have made in the table and record your results in the table. Do you find that θr^2 is approximately constant for all four intervals?
   d) Now you draw in the velocities. Since the orbit is elliptical, the speed is changing. The velocities are labeled v_A, v_B, v_C, v_D, and v_E. For each interval, you set the velocities tail-to-tail and complete the triangle to make the velocity changes Δv_{AB}, Δv_{BC}, Δv_{CD}, and Δv_{DE}. Measure (*) these four velocity changes accurate to the nearest mm and record your results in the table.
   e) According to the inverse square law, the velocity changes Δv measured in (2d) should be inversely proportional to the square of the distances rmeasured in (2b). That is, the product $r^2\Delta v$ ($r \times r \times \Delta v$) should be a constant. Form this product for each of the four arcs using the values you have put in the table and record your result in the table. Do you find that this product is approximately constant?
   f) "That is just the sort of proof I had in mind" exclaims Hooke, "but just tell me one thing. How do you show that the velocity change is always in the direction of the

Sun? If you can't do that, you haven't really demonstrated that the Sun's gravity is responsible for the planet's motion." "That's easy," you reply. You move each Δv back to the planet's orbit, being very careful not to change its direction. (This has already been done for you in Fig. 2 for the first two arcs; you must do it for the last two.) Do you find that the velocity changes Δv point toward the Sun?

You point out to Wren that you have demonstrated that a gravitational force pointing toward the Sun and inversely proportional to the square of the distance from the Sun accounts for Kepler's 3 laws. You respectfully request your reward of 20 pounds sterling. But Halley objects. He says what you have given is only an approximate demonstration and not a proof. "Did you notice", he says to you, "that in your table the value of $r^2\Delta v$ is a bit smaller for arc DE than for the other arcs? This is because the angle θ that the planet travels across DE is so large that its distance from the Sun varies considerably. For such large arcs you cannot approximate the distance by a single number as you have done in your table, but must compute the average. Unfortunately, I do not know how to do this. Alternatively, I am sure that if you made your time intervals shorter and shorter, this defect would gradually disappear. But it would not completely disappear unless you made your intervals infinitely short. In that case, you would not be able to construct your clever triangles and evaluate Δv. Perhaps I will head up to Cambridge next week. There is an eccentric genius named Newton up there who has invented something called 'calculus'. He may be able to solve this problem to my satisfaction."