Astronomy 8824: Problem Set 3

Due Tuesday, October 1, 2019

Root Finding and Minimization

Background Reading: See David's Numerical Methods Notes #3.

Part 1: Root Finding

Plot the functions:

$$\mathrm{f}(x) = 0.25 - 0.6\,x + 0.5\,x^2 - 0.2\,x^3$$

and

$$\mathrm{g}(x) = f(x) + 0.02\ \sin(2\pi x/0.4)$$

over the range $0 \leq x \leq 2$.

Write a code to find the zero of each of these functions by the bisection method.

Apply it to both functions and make plots illustrating the convergence of the method. Among other things, you should plot log(|f(x)|) vs. the number of iterations.

Write a code to find the zero of each of these functions by the Newton-Raphson method. Note that you can easily evaluate the derivatives of these functions analytically. Impose boundaries at $x = 0$ and $x = 2$ on guesses.

Apply it to both functions and make plots illustrating the convergence (or not) of the method.

Comment on the relative strengths and weaknesses of bisection and Newton-Raphson for root-finding.

Part 2. Minimization

For this problem, take slight variations on the previous functions:

$$\mathrm{f}(x) = 0.25 - 0.6\,x + 0.5\,x^2 - 0.2\,x^3 + 0.05\,x^4$$

and

$$\mathrm{g}(x) = f(x) + 0.02\ \sin(2\pi x/0.45)$$

over the range $0 \leq x \leq 2$. Note both the addition of a 4^{th}-order term in *f(x)* and the change of the sine period in *g(x)*. Plot these functions.

Write a code to find the minimum of these functions using the Golden Section Search method. Apply it for several different choices of initial guesses. Make plots that illustrate its performance for the two cases, including log(|x-x_{min}|) and log(|f(x)-f(x_{min})|) where x_{min} is the minimum that you find after convergence.

Part 3. 3-D Minimization

The routine `minimize` from the library `scipy.optimize` can implement the Nelder-Mead algorithm described in NR §10.4 (set `method="nelder-mead"`).

Look up the documentation for this routine and write a short program to minimize the function

$$\mathrm{f}(x_1, x_2, x_3) = x_1^2 + 3x_2^4 + 5x_3^6$$

Try several different starting points with a tolerance of 10^{-6}. Does the routine find the global minimum of this function?

Now minimize the function

$$\mathrm{g}(x_1, x_2, x_3) = (x_1 + 2\sin x_1)^2 + 3(x_2 + 2\sin x_2)^4 + 5(x_3 + 2\sin x_3)^6$$

Try several different starting points with a tolerance of 10^{-6}. Does the routine find the global minimum of this function? Comment.

Part 4. Determining H_0 and Ω_m from the CMB

In Assignment 1, you evaluated the comoving distance to a given redshift for a flat universe with a cosmological constant, ignoring the impact of radiation (which is negligible at low redshift).

The more general expression for comoving distance, allowing space curvature and a dark energy equation of state $w = p/\rho c^2$ assumed to be constant, is

$$D_C(z) = \frac{c}{H_0}\int_0^z \frac{H_0}{H(z')}dz'$$

with

$$\frac{H(z)}{H_0} = \left[\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_k(1+z)^2 + \Omega_{de}(1+z)^{3(1+w)}\right]^{1/2}$$

where

$$\Omega_k = 1 - \Omega_m - \Omega_r - \Omega_{de}$$

The value w=-1 corresponds to a cosmological constant, in which casethe z-dependence of the last term disappears.

In addition to affecting the expansion rate H(z), curvature directly affects the comoving angular diameter distance D_M(z):

$$D_M(z) = \frac{c}{H_0} \frac{\sin\left(\sqrt{\Omega_k} d_C(z)\right)}{\sqrt{-\Omega_k}}$$

where $d_C(z) = D_C(z)/(c/H_0)$.

To interpret this notation for positive Ω_k, recall that sin(ix) = *i* sinh *x*.

To save you some work, you may use David's code `cosmodist.py` and `cosmodist_subs.py` from his web page, which compute the comoving and angular diameter distance for specified H_0, Ω_m, Ω_k, and w.

The notation Ω_x represents the ratio of the energy density of component *x* to the critical density required for a flat universe. Because the critical density is proportional to H_0^2, physical densities are proportional to $\Omega_x h^2$ where

$$h \equiv \frac{H_0}{100 \text{ km s}^{-1}\text{Mpc}^{-1}}$$

is a convenient dimensionless scaling of the Hubble constant.

The value of $\Omega_r h^2 = 4.183 \times 10^{-5}$ is known from the temperature of the CMB and standard early universe neutrino physics.

Modeling the Planck CMB power spectrum gives high-precision constraints on

$$\Omega_m h^2 \approx 0.1386$$

and

$$D_M(z = 1090) \approx 13960 \text{ Mpc}$$

Take these two measurements to be exact, i.e., with no uncertainty.

Modify your bisection root finding routine so that instead of calling a generic function it calls the `cosmodist` function and finds the value of H_0 that satisfies the above constraints, assuming a flat universe with a cosmological constant ($\Omega_k = 0$, w = -1).

(a) What are the values of H_0 and Ω_m that satisfy the above equations?

(b) Why did we need to include radiation in our calculation even though $\Omega_r h^2$ is only ~4 x 10^{-5}?

(c) Suppose we allowed $\Omega_k \neq 0$, w $\neq$ -1. Would the two CMB constraints that you used here still suffice to determine H_0 and Ω_m? Why or why not?

Note: This problem set was developed by David Weinberg.