Astronomy 871, Autumn 2008, Problem Set 1

Due Wed, Oct 8 in class

Problem 1:

Show that the "Field Criterion" for thermal stability:

$$\left(\frac{\partial L}{\partial T}\right)_P < 0$$

reduces to the condition

$$\left(\frac{\partial \ln \Lambda}{\partial \ln T}\right)_P < 1$$

for the case of isobaric perturbations in a perfect gas subjected to a constant heating rate G. This was the proof left as a problem in Chapter 1 of the notes.

Problem 2:

This problem takes you through a simplified version of the Field, Goldsmith, & Habing (1969) 2-phase ISM analysis. Here, we define the following terms:

n : the total atomic density $n = (n_H + n_p)$

$n_e = x_e n$: electron density expressed in terms of the fractional ionization, x_e

$\alpha_B(T)$: Case B recombination coefficient for Hydrogen, in units of cm^3 s^{-1}

The problem has 6 parts:

a) The volumetric cooling rate, $n^2\Lambda(x_e,T)$ consists of 2 parts:

[CII] 158μm line cooling: $n_{C^+} n_e \Lambda_C(T) = n_e n \Lambda'_C(T)$

where $\Lambda'_C(T)$ includes the abundance of C^+ relative to *total* H (n_H+n_p).

Lyα Cooling: $n_e n_H \Lambda_H(T)$

The total volumetric cooling rate is

$$n^2\Lambda(x_e,T) = n_{C^+} n_e \Lambda_C(T) + n_e n_H \Lambda_H(T)$$

Rework this into an expression for $\Lambda(x_e,T)$ as a function of $\Lambda'_C(T)$, $\Lambda_H(T)$, and x_e, eliminating the densities using the definitions of n and n_e above.

b) The volumetric heating rate due to cosmic rays is $nG = n_H \zeta_{CR} \varepsilon_H$, where ζ_{CR} is the cosmic ray ionization rate per atom (a constant), and ε_H is the mean electron energy, ~50eV. Balance the rate of primary CR ionization, $n_H \zeta_{CR}$, by recombination, $n_e n_p \alpha_B(T)$ and show that this condition reduces to:

$$\zeta_{CR}(1 - x_e) = n x_e^2 \alpha_B(T)$$

c) Show that the condition of thermal balance, $n^2\Lambda - nG = 0$, reduces to a relation of the form

$$\frac{n}{\zeta_{CR}} = \frac{\varepsilon_H(1-x_e)}{\Lambda(x_e,T)}$$

Substitute this into the ionization equilibrium condition from part (b) to derive this relation:

$$x_e^2 = \frac{\Lambda(x_e,T)}{\varepsilon_H \alpha_B(T)}$$

Substitute in our definition of $\Lambda(x_e,T)$, and solve this to give an expression for x_e as a function of T, (but keep the $\Lambda'_C(T)$ and $\Lambda_H(T)$ terms intact for now to avoid making a mess of the algebra).

d) You are given the following approximate numerical forms of $\alpha_B(T)$, $\Lambda'_C(T)$ and $\Lambda_H(T)$:

$$\alpha_B(T) \approx 1.56\times10^{-12}\ \text{cm}^3\ \text{s}^{-1} \begin{cases} (T/1000K)^{-0.65} & \text{for T} \le 1000\text{K} \\ (T/1000K)^{-0.80} & \text{for T} \ge 1000\text{K} \end{cases}$$

$$\Lambda'_C(T) \approx 2.4\times10^{-24} T^{-1/2} e^{-92/T}$$

$$\Lambda_H(T) \approx 9.2\times10^{-17} T^{1/2}\left(1+\frac{17500}{T}\right) e^{-118,400/T}$$

The formula for $\alpha_B(T)$ is valid for $1.0 \le \log T \le 4.5$. Note that these formulae will differ in detail from formulae given in the notes (we're reproducing an historical derivation).

Using the formulae you have derived in the previous steps, write a simple program/script/SM macro (you choose) to compute the following quantities for log T in the range 1.0 to 4.4 in steps of $\Delta \log T = 0.02$

i. x_e
ii. (n/ζ_{CR})
iii. $(\zeta_{CR} k/P) = (G/nT)$, where $P = (n+n_e)kT = n(1+x_e)kT$.

and make plots of $\log(\zeta_{CR} k/P)$ and $\log(x_e)$ as functions of $\log T$ (the reasons for this seemingly peculiar ordering will become obvious when you start coding.) The logarithmic step size was chosen to ensure that you will sample rapid changes in these quantities. Note that we do not need to assume an explicit value for ζ_{CR} at this point, which is what all the proceeding algebraic tap-dancing was all about! In fact, we will estimate ζ_{CR} after the fact from observations.

e) In your $\log(\zeta_{CR} k/P)$ plot, identify the 2-phase region and draw a line of constant pressure for a representative locus through the middle of this region, labeling the FGH points like we did in the lecture notes. What are the temperatures of the F, G, and H points you have chosen?

f) Pulsar dispersion-measure observations yield a mean electron density in the ISM of $0.03\ \text{cm}^{-3}$. From your diagrams and calculations, estimate the CR heating rate, ζ_{CR}, required to ensure the existence of two stable warm and cold phases of the ISM? [Hint: use your program to calculate x_e and (n/ζ_{CR}) for the log(T) you estimate for your "F" point in your 2-phase diagram, you known $\langle n_e \rangle$ ...]. How does this compare with the value $\zeta_{CR} = 10^{-15} s^{-1}$ usually assumed by earlier versions of the model? Comment on the agreement (or lack thereof) in light of what we said in class about heating and cooling the general ISM.