ASTRONOMY 294Z: The History of the Universe
Professor Barbara Ryden

SOLUTIONS TO PROBLEM SET # 4

1) [20 points] *Today, the average density of matter in the universe is* $\rho = 3 \times 10^{-27}\,\mathrm{kg/m^3}$. *If the matter consisted entirely of hydrogen atoms, how many hydrogen atoms (on average) would be contained in a cubic meter of the universe? If the matter consisted entirely of regulation baseballs (of mass* $M = 0.145\,\mathrm{kg}$ *apiece), how many baseballs (on average) would be contained in a cubic astronomical unit of the universe?*

The mass of a single hydrogen atom is $m = 1.7 \times 10^{-27}\,\mathrm{kg}$. In order for the mass density to be $\rho = 3 \times 10^{-27}\,\mathrm{kg/m^3}$, the number of hydrogen atoms per cubic meter would have to be

$$n = \frac{\rho}{m} = \frac{3 \times 10^{-27}\,\mathrm{kg/m^3}}{1.7 \times 10^{-27}\,\mathrm{kg}} = 1.76/\mathrm{m}^3 \ . \qquad (1)$$

Fewer than two atoms, on average, per cubic meter of space – that's a very low density.

The mass of a single baseball is $M = 0.145\,\mathrm{kg}$. In order for the mass density to be $\rho = 3 \times 10^{-27}\,\mathrm{kg/m^3}$, the number of baseballs per cubic AU would have to be

$$n = \frac{\rho}{M} = \left(\frac{3 \times 10^{-27}\,\mathrm{kg/m^3}}{0.145\,\mathrm{kg}}\right)\left(\frac{1.5 \times 10^{11}\,\mathrm{m}}{1\,\mathrm{AU}}\right)^3 = 69{,}800{,}000/\mathrm{AU}^3 \ . \qquad (2)$$

2) [20 points] *The temperature of the cosmic background light today is* $T \approx 3\,\mathrm{K}$. *At the time the universe became transparent, the temperature of the cosmic background light was* $T \approx 3000\,\mathrm{K}$. *This means that the universe has expanded by a factor of 1000 since it became transparent. If the density of matter today is* $\rho = 3 \times 10^{-27}\,\mathrm{kg/m^3}$, *what was the density of matter when the universe became transparent?*

Consider a cube that is expanding along with the general expansion of the universe. The edges of the cube currently have a length $\ell_{\rm now} = 1\,\mathrm{m}$. The volume of the cube is $\ell_{\rm now}^3 = 1\,\mathrm{m}^3$, and thus the mass it contains is $M = \rho_{\rm now}\ell_{\rm now}^3 = 3 \times 10^{-27}\,\mathrm{kg}$. Since matter is not created or destroyed, the mass M inside the cube remains constant as the universe expands. At the time the universe became transparent, the expanding cube had sides of length

$$\ell_{\rm trans} = \frac{\ell_{\rm now}}{1000} = \frac{1\,\mathrm{m}}{1000} = 0.001\,\mathrm{m}\ . \tag{3}$$

The density within the box when the universe became transparent was

$$\rho_{\rm trans} = \frac{M}{\ell_{\rm trans}^3} = \frac{3 \times 10^{-27}\,\mathrm{kg}}{(0.001\,\mathrm{m})^3} = \frac{3 \times 10^{-27}\,\mathrm{kg}}{10^{-9}\,\mathrm{m}^3} = 3 \times 10^{-18}\,\mathrm{kg/m^3}\ . \tag{4}$$

If lengths in the universe increase by a factor of one thousand, then the density of matter decreases by a factor of one billion.

3) [20 points] *At the time of primordial nucleosynthesis, the temeprature of the cosmic background light was* $T \approx 4.8 \times 10^8\,\mathrm{K}$*. By what factor has the universe expanded since the time of primordial nucleosynthesis? If the density of matter today is* $\rho = 3 \times 10^{-27}\,\mathrm{kg/m^3}$*, what was the density of matter at the time of primordial nucleosynthesis?*

The temperature at the time of primordial nucleosynthesis was $T_{\mathrm{nuc}} = 4.8 \times 10^8\,\mathrm{K}$. The temperature now is $T_{\mathrm{now}} = 3\,\mathrm{K}$. The universe has expanded by a factor of

$$\frac{T_{\mathrm{nuc}}}{T_{\mathrm{now}}} = \frac{4.8 \times 10^8\,\mathrm{K}}{3\,\mathrm{K}} = 1.6 \times 10^8 \tag{5}$$

since the time of primordial nucleosynthesis. That is, a cube that currently has sides of length $\ell_{\mathrm{now}} = 1\,\mathrm{m}$ had sides of length

$$\ell_{\mathrm{nuc}} = \frac{\ell_{\mathrm{now}}}{1.6 \times 10^8} = \frac{1\,\mathrm{m}}{1.6 \times 10^8} = 6.25 \times 10^{-9}\,\mathrm{m}\ . \tag{6}$$

Using same argument as in the previous problem, we deduce that the matter density at the time of primordial nucleosynthesis was

$$\rho_{\mathrm{nuc}} = \frac{M}{\ell_{\mathrm{nuc}}^3} = \frac{3 \times 10^{-27}\,\mathrm{kg}}{(6.25 \times 10^{-9}\,\mathrm{m})^3} = 0.012\,\mathrm{kg/m^3}\ . \tag{7}$$

4) [20 points] *Look up the density of the Earth's air at sea level. Is the density of matter at the time of primordial nucleosynthesis greater than or less than the density of the Earth's air at sea level?*

According to the Wikipedia article "Density of Air", dry air has a density $\rho_{\rm air} \approx 1.2\,{\rm kg}/\,{\rm m}^3$ at sea level at a temperature $T = 20°C = 68°F$. (It becomes denser when cooler, and less dense when warmer.) Thus, the density of matter at the time of primordial nucleosynthesis is *less than* the density of the Earth's air at sea level, by a factor of a hundred. (Although the universe started out in an extraordinarily dense state, by the time several minutes have passed since the Big Bang, the density of matter has dropped to something that's not outlandishly high, by terrestrial standards.)

5) [20 points] *Suppose that you have used a Cepheid variable star as a "standard candle" to compute the distance to a particular galaxy. The distance computed is* $d = 35\,\mathrm{Mpc}$*. Much to your embarrassment, you find that the Cepheid variable star has a luminosity* L *that is actually* twice *the luminosity you assumed when making your calculation. Is the galaxy* closer *or* farther *than you originally calculated? What is the true distance to the galaxy?*

For a standard candle of assumed luminosity L and measured flux F, the distance is

$$d = \sqrt{\frac{L}{4\pi F}} \ . \tag{8}$$

When you increase your assumed luminosity L, you increase the computed distance; thus, the galaxy is *farther* than you originally calculated.

Originally, using a false luminosity L_{false}, you computed

$$d_{\mathrm{false}} = \sqrt{\frac{L_{\mathrm{false}}}{4\pi F}} = 35\,\mathrm{Mpc} \ . \tag{9}$$

However, the true distance, using the correct luminosity $L_{\mathrm{true}} = 2L_{\mathrm{false}}$, is

$$d_{\mathrm{true}} = \sqrt{\frac{L_{\mathrm{true}}}{4\pi F}} = \sqrt{\frac{2L_{\mathrm{false}}}{4\pi F}} = \sqrt{2} \times \sqrt{\frac{L_{\mathrm{false}}}{4\pi F}} \ . \tag{10}$$

Comparing equation (10) with equation (9), we find that

$$d_{\mathrm{true}} = \sqrt{2} \times d_{\mathrm{false}} = \sqrt{2} \times 35\,\mathrm{Mpc} = 49.5\,\mathrm{Mpc} \ . \tag{11}$$