AST 822: Electromagnetic Radiation

Problem Set 6: due Wednesday, November 23

1) A highly relativistic electron initially has Lorentz factor $\gamma_0 \gg 1$. It is moving in a magnetic field of constant flux density $\vec{B}$ with a pitch angle α. What is $\gamma(t)$ as the electron radiates away its energy in the form of synchrotron radiation? What is $t_{1/2}$, the time required for the electron to radiate away half its initial energy? If $\alpha = \pi/2$ and $B = 10^{-6}\,$gauss (a typical interstellar value), for what values of γ_0 is $t_{1/2} \leq 10\,\mathrm{Gyr}$ (the approximate age of the Galaxy)?

2) A high energy cosmic ray proton is emitted at the center of the Galaxy, then starts to orbit in the Galaxy's magnetic field, of strength $B = 10^{-6}\,$gauss.
a) If the proton is on a circular orbit, what is the orbital radius as a function of the proton energy ϵ? At what energy is the orbital radius equal to the distance of the Sun from the Galactic center?
b) If the Galactic center emitted cosmic ray protons with energy $\epsilon = 10^{21}\,\mathrm{eV}$, and if $\vec{B}$ were constant between the Sun and the Galactic center, what would be the maximum angular deflection θ you could observe for a proton arriving at the Sun's location?

[Continued on back]

3) The Crab nebula, which currently has a radius $R = 1.8\,\mathrm{pc}$, was created by the supernova explosion of the year AD 1054. The light from the nebula, which is highly polarized, comes primarily from synchrotron radiation. The spectrum of the nebula is reasonably well fit by a broken power law. At $\nu < 10^{13}\,\mathrm{Hz}$, the intensity is $I_\nu \propto \nu^{-s}$, with $s = 0.27$. At $\nu > 10^{13}\,\mathrm{Hz}$, the index steepens to $s = 0.77$.

a) Suppose that starting at a time $t = 0$, relativistic electrons have been injected into the nebula at the constant rate $S(\gamma) \propto \gamma^{-p}$, where γ is the Lorentz factor. That is, $S(\gamma)d\gamma$ is the rate, in units of electrons per second, at which electrons in the range $\gamma \to \gamma + d\gamma$ are injected into the nebula. Assume that the Crab nebula has a constant magnetic flux density B. What is $N(\gamma, t)$, where $N(\gamma, t)d\gamma$ is the total number of electrons in the range $\gamma \to \gamma + d\gamma$ at time t? (Hint: use the results from problem 1; for simplicity, you may assume that all electrons have a pitch angle $\alpha = \pi/2$.)

b) Let $\gamma_{1/2}$ be the value of γ for which $t_{1/2}$ (see problem 1) is equal to the observed age of the Crab nebula. Show that the "break" in the power spectrum should occur at a frequency

$$\nu_{\mathrm{break}} \approx \frac{Be}{m_e c}\gamma_{1/2}^2 \; . \tag{1}$$

Given that $\nu_{\mathrm{break}} \approx 10^{13}\,\mathrm{Hz}$, what is the value of B in the Crab nebula? What is the value of p?

c) What is the energy density U_e of the relativistic electrons in the Crab nebula? What is the total magnetic energy density, U_B? What is the total energy, in ergs, of the relativistic electrons and the magnetic field combined?