Astronomy 825:
Radiative Gas Dynamics
Winter 2009

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Contents

Notes on the Notes

These lecture notes are the result of my having taught Astronomy 825 several times, starting in the 1992/93 academic year, when my knowledge of radiative gas dynamics was nearly non-existent. The most valuable resource I found while frantically cramming to teach the course was the two-volume *Physics of Astrophysics* set by Frank Shu. Volume I of Shu's work is titled *Radiation* and Volume II is titled *Gas Dynamics*, and between them they cover more than you will probably want to know about radiative gas dynamics. More recent works that I have consulted while updating these notes are *An Introduction to Astrophysical Fluid Dynamics* by Michael Thompson and *Principles of Astrophysical Fluid Dynamics* by C. J. Clarke and R. F. Carswell.

In one way, these (not-a-textbook) notes are a sequel to Rick Pogge's (not-a-textbook) notes for Astronomy 871 (*Physics of the Interstellar Medium*). I have made an attempt to make notation and units consistent between the two sets of notes. Thus, I adopt cgs units, supplemented by angstroms (Å), electron volts (eV), and parsecs (pc).

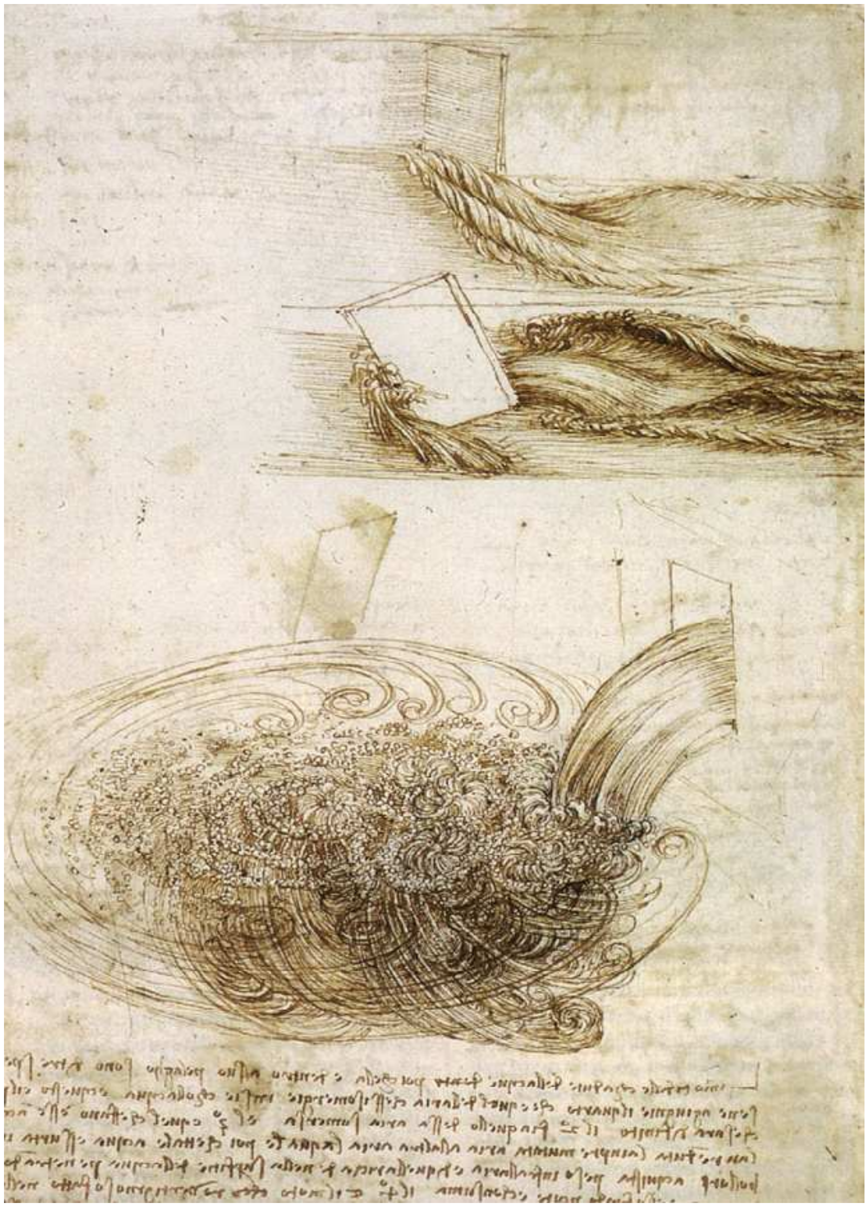

Figure 1: Turbulent fluids as drawn by Leonardo da Vinci.

Chapter 1

Fundamentals of Gas Dynamics

Gas dynamics is the study of continuous compressible fluids in motion.[1] Real gases, of course, are made of individual particles: atoms, molecules, ions, and/or electrons. However, a gas can be adequately approximated as a continuous fluid when

$$\lambda \ll L \tag{1.1}$$

where λ is the mean free path of the gas particles, and L is the characteristic size of the system. The mean free path in a gas of neutral particles is

$$\lambda = \frac{1}{n\sigma} \tag{1.2}$$

where n is the number density of particles, and σ is the cross section for collisions. A typical cross section for neutral atoms is $\sigma \sim 10^{-15}\,\mathrm{cm}^2$. In air at room temperature, $n \sim 10^{19}\,\mathrm{cm}^{-3}$, and hence $\lambda \sim 10^{-4}\,\mathrm{cm}$. Thus, a volume of air that is larger than several microns on a side can be treated as a continuous fluid. In the interstellar medium (ISM), as you learned in Ast 871, particle number densities are much lower than in the Earth's atmosphere. In a molecular cloud, $n \sim 1000\,\mathrm{cm}^{-3}$, and hence $\lambda \sim 10^{12}\,\mathrm{cm} \sim 0.07\,\mathrm{AU}$. In the warm neutral medium, $n \sim 0.5\,\mathrm{cm}^{-3}$, and hence $\lambda \sim 2 \times 10^{15}\,\mathrm{cm} \sim 100\,\mathrm{AU} \sim 6 \times 10^{-4}\,\mathrm{pc}$.

The situation is more complicated in a plasma. Consider a gas of fully ionized hydrogen. The effective radius of interaction r_e for a free electron

[1] *Hydrodynamics*, which you might guess from its etymology to be the study of water in motion, is actually the study of continuous fluids in motion, and can embrace both compressible fluids (gas) and incompressible fluids (liquid).

can be found by setting the magnitude of its potential energy at a distance r_e from an electron or proton equal to its thermal kinetic energy: is found by setting

$$\frac{e^2}{r_e} \sim m_e v_e^2 \ . \tag{1.3}$$

Since $m_e v_e^2 \sim kT$, where T is the kinetic temperature of the free electrons, we can write

$$r_e \sim \frac{e^2}{m_e v_e^2} \sim \frac{e^2}{kT} \ . \tag{1.4}$$

The cross section is thus

$$\sigma \sim \pi r_e^2 \sim \frac{\pi e^4}{k^2 T^2} \ , \tag{1.5}$$

and the mean free path for an electron is

$$\lambda \sim \frac{k^2 T^2}{\pi e^4 n} \ . \tag{1.6}$$

(Note: a more accurate calculation contains the Coulomb logarithm $\ln \Lambda$, but this is good enough for an order-of-magnitude estimate.) In the hot ionized interstellar medium, $T \sim 10^6\,\mathrm{K}$ and $n \sim 3 \times 10^{-3}\,\mathrm{cm}^{-3}$. The mean free path is thus $\lambda \sim 4 \times 10^{19}\,\mathrm{cm} \sim 10\,\mathrm{pc}$.

It's useful at this point to provide a thumbnail sketch of the different components of the ISM. **Molecular Clouds** consist mainly of molecular gas (H_2, CO, He, etc.) Typical temperatures and number densities are

$$T \sim 15\,\mathrm{K} \text{ and } n \gtrsim 1000\,\mathrm{cm}^{-3} \ . \tag{1.7}$$

The **Cold Neutral Medium** consists mainly of neutral atomic gas (H I, He I, etc.) Typical temperatures and number densities are

$$T \sim 90\,\mathrm{K} \text{ and } n \sim 50\,\mathrm{cm}^{-3} \ . \tag{1.8}$$

The **Warm Neutral Medium** consists mainly of neutral atomic gas, but at a higher temperature and lower density than the Cold Neutral Medium. Typical temperatures and number densities are

$$T \sim 8000\,\mathrm{K} \text{ and } n \sim 0.5\,\mathrm{cm}^{-3} \ . \tag{1.9}$$

The **Warm Ionized Medium** consists of partially ionized gas (H I, He I, H II, He II, He III, etc.) Typical temperatures and number densities are

$$T \sim 10^4\,\mathrm{K} \text{ and } n \sim 0.1\,\mathrm{cm}^{-3} \ . \tag{1.10}$$

The **Hot Ionized Medium** consists of ionized gas at very high temperatures. Typical temperatures and number densities are

$$T \gtrsim 10^6\,\mathrm{K} \text{ and } n \lesssim 0.003\,\mathrm{cm}^{-3}\ . \tag{1.11}$$

Although the different components of the ISM are not in perfect pressure equilibrium, it is notable that the pressures implied by the values of n and T given above are all within an order of magnitude of each other, with a typical value

$$P = nkT \sim 3 \times 10^{-13}\mathrm{dyne\ cm}^{-2} \sim 3 \times 10^{-19}\,\mathrm{atm}\ . \tag{1.12}$$

The "atmosphere" (atm) is approximately equal to the pressure of the Earth's atmosphere at sea level. Since $1\,\mathrm{atm} \approx 10^6\mathrm{dyne\ cm}^{-2}$, the ISM is obviously very low in pressure compared to the air around you.

Some regions of the ISM are emphatically *not* in pressure equilibrium. The densest cores of molecular clouds, for instance, are self-gravitating systems that are, on occasion, unstable to collapse and fragmentation into protostars. H II regions, with $n \gtrsim 10\,\mathrm{cm}^{-3}$ and $T \sim 10^4\,\mathrm{K}$, are generally in expansion. The same is true of supernova remnants, which have $n \gtrsim 1\,\mathrm{cm}^{-3}$ and temperatures up to $10^7\,\mathrm{K}$.

Much of the ISM is flowing out of, or accreting onto, compact objects. Examples of **outflow** are blast waves, winds, and jets. Examples of **accretion** are cooling flows and accretion disks. Interesting things also happen where one phase of the interstellar medium meets another. A **shock front** is a boundary between low density, supersonic flow and high density, subsonic flow. An **ionization front** is a boundary between ionized matter and neutral matter.

In many astrophysical contexts, radiative effects strongly influence the flow of gas. **Molecular Clouds** can absorb and emit radiation by rotational and vibrational transitions. Sufficiently energetic radiation can photodissociate the molecules. The dissociation energy of H_2 is 4.48 eV, corresponding to a wavelength of 2770Å. **Neutral Atomic Gas** can absorb and emit radiation through bound-bound electronic transitions. Sufficiently energetic radiation can photoionize the atoms. The ionization energy of H is 13.6 eV, corresponding to a wavelength of 912Å. **Ionized Gas** can emit radiation through bremsstrahlung (free-free transitions) and radiative recombination. The electrons and ions can also interact with photons via Compton scattering. In the presence of a magnetic field, electrons can lose energy by synchrotron radiation.

In Astro 873 (Interstellar Medium), you learned a great deal about radiative transfer in the interstellar medium. It wasn't always pretty. To make life simpler for ourselves, let's start by considering the dynamics of gas that is *not* radiating. A good place to start is with the **Boltzmann Equation**.[2] At a given time t, any particle of the gas has a position $\vec{x}$ and a velocity $\vec{v}$. So, if we live in a Newtonian deterministic universe, we can compute the development of a system if we know all the positions and velocities at a given time and if we know the external forces (gravitational, electrostatic, magnetic, etc.) working on the particles. This fully deterministic approach, however, becomes a major pain in the neck as the number of gas particles becomes large. If you are dealing with many, many, many gas particles, a probabilistic approach becomes much more practical ... and also much more useful.

Let $f(\vec{x},\vec{v},t)d^3xd^3v$ be the probability of finding a gas particle within the six-dimensional phase space volume d^3xd^3v at position $\vec{x}$ with a velocity $\vec{v}$ at time t. The six-dimensional space mapped out by $\vec{x}$ and $\vec{v}$ is known as **phase space**. The three-dimensional space mapped by $\vec{x}$ is known as **position space**; the three-dimensional space mapped by $\vec{v}$ is known as **velocity space**. The function $f(\vec{x},\vec{v},t)$ is known as the **distribution function**.

The distribution function is normalized so that

$$\int\int f(\vec{x},\vec{v},t)d^3xd^3v = N \ , \tag{1.13}$$

where N is the total number of particles in the system. If we choose, we can also define a *mass* distribution function

$$f_m(\vec{x},\vec{v},t) = mf(\vec{x},\vec{v},t) \ , \tag{1.14}$$

where m is the mass of an individual particle. If n types of particle are present, each with a different mass, the *total* mass distribution function is

$$f_m = \sum_{k=1}^{n} m_k f_k(\vec{x},\vec{v},t) \ , \tag{1.15}$$

where m_k is the mass of the k^{th} type of particle. Since particles are neither created nor destroyed,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\dot{x}_i\frac{\partial f}{\partial x_i} + \dot{v}_i\frac{\partial f}{\partial v_i}\right) = \frac{df}{dt}\Big|_c \ . \tag{1.16}$$

[2] Pay close attention to the Boltzmann Equation; it also plays a key role in the dynamics of stellar systems.

The factor on the right hand side of the above equation is the rate at which particles are bumped into a phase space volume by collisions with other particles.[3]

We may rewrite equation (1.16) by making use of the fact that $\dot{\vec{x}} = \vec{v}$ and that $\dot{\vec{v}} = \vec{g}$, where $\vec{g}$ is the acceleration of a particle at position $\vec{x}$ with velocity $\vec{v}$. The gravitational force acting on the particle results in an acceleration

$$\vec{g} = -\vec{\nabla}\Phi \ , \tag{1.17}$$

where the potential $\Phi(\vec{x})$ has contributions both from the self-gravity of the system of particles and from any external gravitational fields that may be present.[4] The electromagnetic force acting on the particle results in an acceleration

$$\vec{g} = \frac{q}{m}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \ , \tag{1.18}$$

where q is the charge of of the particle, m is its mass, $\vec{E}$ is the electric field at the particle's location, and $\vec{B}$ is the magnetic field there.

The phase space continuity equation can now be written in the form

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(v_i \frac{\partial f}{\partial x_i} + g_i \frac{\partial f}{\partial v_i}\right) = \frac{df}{dt}\Big|_c \ . \tag{1.19}$$

This equation is known as the **Boltzmann equation**.

In equilibrium, particles are bumped into an element of phase space at the same rate that they are bumped out, and the collisional term on the right-hand side is equal to zero. If the collisional term is ignored, the resulting equation is called the **collisionless Boltzmann equation**.

The collisionless Boltzmann equation is the basis of stellar dynamics and of hydrodynamics. However, we are generally unable to determine the full

[3] The word "collisions" can be misleading, since it implies that the individual particles behave like billiard balls, only interacting when they are in contact with each other. In an ionized gas, the individual charged particles interact through long-range electromagnetic forces. A "collision", in that case, is a close encounter between charged particles during which the electromagnetic force changes their direction of motion by $\sim 90°$ or more. Similarly, in stellar dynamics, a "collision" between stars is a close encounter during which gravity changes their direction of motion by $\sim 90°$ or more.

[4] These notes are highly Newtonian. Unless I make an explicit statement to the contrary, all motions are assumed to be non-relativistic, and gravity is assumed to be an inverse-square-law force.

velocity distribution at every point in position space. What we usually want to deal with is the number density or mass density in position space, plus the mean (or bulk) velocity of the gas, plus its velocity dispersion.

Now, let's take the moments of the collisionless Boltzmann equation. Let's start with f_m (the mass density in phase space) rather than f (the number density in phase space), because it is useful to think about the conservation of mass (m) , the conservation of momentum ($m \times \vec{v}$), and the conservation of kinetic energy ($m \times v^2/2$). The mass density in position space is found by integrating over all velocities:

$$\rho(\vec{x}, t) = \int f_m(\vec{x}, \vec{v}, t) d^3v \ . \tag{1.20}$$

For any measurable quantity Q, the mass-weighted average value at a position $\vec{x}$ at time t is

$$\langle Q \rangle = \frac{1}{\rho} \int Q f_m d^3v \ . \tag{1.21}$$

Integrate the collisionless Boltzmann equation over all velocities:

$$\frac{\partial}{\partial t} \int f d^3v + \sum_i \frac{\partial}{\partial x_i} \int v_i f d^3v = -\sum_i g_i \int \frac{\partial f}{\partial v_i} d^3v \ . \tag{1.22}$$

(The distribution function that I am using is f_m; the subscript is suppressed to avoid clutter.) The partial differentials have been taken outside the integrals since $\vec{x}$, $\vec{v}$, and t are all independent variables. The right hand side of the above equation vanishes by the divergence theorem, as long as $f \to 0$ as $v \to \infty$.

Then

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial}{\partial x_i}(\rho \langle v_i \rangle) = 0 \ . \tag{1.23}$$

Now introduce a new symbol:

$$\vec{u} \equiv \langle \vec{v} \rangle \ . \tag{1.24}$$

The velocity $\vec{u}$ is the **bulk velocity** at a given point in space. The velocity $\vec{v}$ of a particular particle may then be broken into two components:

$$\vec{v} = \vec{u} + \vec{w} \ , \tag{1.25}$$

where $\vec{w}$ is the **random velocity**.

In vector form,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \ . \tag{1.26}$$

This is the **continuity equation**. It states that mass is conserved. Moreover, the flow of matter is continuous; mass does not disappear at one point and simultaneously appear at another point some distance away.

The continuity equation is a single equation in four unknowns (ρ and the three components of $\vec{v}$). So, let's find the next higher moment of the collisionless Boltzmann equation; multiply by v_j and integrate over all velocities.

$$\frac{\partial}{\partial t} \int v_j f d^3v + \sum_i \frac{\partial}{\partial x_i} \int v_j v_i f d^3v = -\sum_i g_i \int v_j \frac{\partial f}{\partial v_i} d^3v \ . \tag{1.27}$$

Rewriting the integrand on the right hand side, we have

$$\frac{\partial}{\partial t}(\rho \langle v_j \rangle) + \sum_i \frac{\partial}{\partial x_i}(\rho \langle v_j v_i \rangle) = -\sum_i g_i \int \left[\frac{\partial}{\partial v_i}(v_j f) - \delta_{ij} f \right] d^3v \ . \tag{1.28}$$

Now we use the decomposition $\vec{v} = \vec{u} + \vec{w}$ on the left, and the divergence theorem on the right (assuming that $fv \to 0$ as $v \to \infty$). This tells us that

$$\frac{\partial}{\partial t}(\rho u_j) + \sum_i \frac{\partial}{\partial x_i}(\rho u_i u_j + \rho \langle w_i w_j \rangle) = \rho g_j \ . \tag{1.29}$$

This is the equation of momentum conservation. It is convenient to write the pressure tensor $\rho \langle w_i w_j \rangle$ in the form

$$\rho \langle w_i w_j \rangle = P \delta_{ij} - \pi_{ij} \ , \tag{1.30}$$

where P is the **pressure**,

$$P \equiv \frac{1}{3} \rho \langle |\vec{w}|^2 \rangle \ , \tag{1.31}$$

and π_{ij} is the **viscous stress tensor**

$$\pi_{ij} \equiv P \delta_{ij} - \rho \langle w_i w_j \rangle \ . \tag{1.32}$$

It is useful to make this division because in some cases the viscous stress tensor can be ignored. In other cases, the viscous stress tensor can be computed in terms of the the shear of the bulk velocity.

The equation for momentum conservation using the formalism that we have adopted:

$$\frac{\partial}{\partial t}(\rho u_j) + \sum_i \frac{\partial}{\partial x_i}(\rho u_i u_j + P\delta_{ij} - \pi_{ij}) = \rho g_j \ . \tag{1.33}$$

The equation can be written in tensor form:

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u}\vec{u} + P \overleftrightarrow{I} - \overleftrightarrow{\pi}) = \rho \vec{g} \ . \tag{1.34}$$

In the above equation, $\overleftrightarrow{I}$ is the unit matrix:

$$\overleftrightarrow{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{1.35}$$

The tensor product of the velocities is

$$\vec{u}\vec{u} = \begin{pmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{pmatrix} \ . \tag{1.36}$$

Finally, the viscous stress tensor is the symmetric traceless tensor

$$\overleftrightarrow{\pi} = \begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{12} & \pi_{22} & \pi_{23} \\ \pi_{13} & \pi_{23} & -\pi_{11} - \pi_{22} \end{pmatrix} \ . \tag{1.37}$$

The tensor form of the momentum conservation equation tells us that the time derivative of a conserved quantity plus the divergence of a flux is equal to a source term; the standard form of a conservation equation.

Let's go one step further and look at the conservation of kinetic energy. Multiply the collisionless Boltzmann equation by v^2, and integrate over all velocities.

$$\frac{\partial}{\partial t}\int v^2 f d^3v + \sum_i \frac{\partial}{\partial x_i}\int v_i v^2 f d^3v = -\sum_i g_i \int v^2 \frac{\partial f}{\partial v_i} d^3v \ . \tag{1.38}$$

Rewriting the integrand on the right hand side,

$$\frac{\partial}{\partial t}(\rho\langle v^2\rangle) + \sum_i(\rho\langle v_i v^2\rangle) = -\sum_i g_i \int \left[\frac{\partial}{\partial v_i}(v^2 f) - 2v_i f\right] d^3v \ . \tag{1.39}$$

Breaking the velocity into its ordered and random components ($\vec{v} = \vec{u} + \vec{w}$) we find

$$\frac{\partial}{\partial t}(\rho u^2 + \rho\langle w^2\rangle) + \sum_i \frac{\partial}{\partial x_i}(\rho[u_i u^2 + u_i\langle w^2\rangle + 2\sum_j u_j\langle w_i w_j\rangle + \langle w_i w^2\rangle]) = 2\sum_i g_i \rho u_i \ . \tag{1.40}$$

Some more definitions: the **specific internal energy** of a monatomic gas is

$$\varepsilon = \frac{1}{2}\langle w^2\rangle \ ; \tag{1.41}$$

this is just the energy per unit mass contributed by the random internal motions. Note that for a monatomic gas

$$\varepsilon = \frac{3}{2}\frac{P}{\rho} \ . \tag{1.42}$$

Next, the **conduction heat flux** is

$$\vec{F} \equiv \frac{1}{2}\rho\langle \vec{w} w^2\rangle \ . \tag{1.43}$$

If the distribution of random velocities $\vec{w}$ is symmetric about zero, then the conduction heat flux $\vec{F}$ will vanish. However, if $\vec{w}$ has a skewed distribution, then the "hot" particles (those with large random velocities) will have a net drift relative to the "cold" particles.

Using the newly defined quantities, the equation of energy conservation is

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho u^2 + \rho\varepsilon) + \vec{\nabla}\cdot\left[(\frac{1}{2}\rho u^2 + P + \rho\varepsilon)\vec{u} - \overleftrightarrow{\pi}\cdot\vec{u} + \vec{F}\right] = \rho\vec{u}\cdot\vec{g} \ . \tag{1.44}$$

By combining the mass conservation equation with the momentum conservation equation, we can write down an equation for the conservation of the bulk kinetic energy ($\rho u^2/2$):

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho u^2) + \vec{\nabla}\cdot(\frac{1}{2}\rho u^2\vec{u}) = \rho\vec{u}\cdot\vec{g} - \vec{u}\cdot\vec{\nabla}P + \vec{u}\cdot(\vec{\nabla}\cdot\overleftrightarrow{\pi}) \ . \tag{1.45}$$

By subtracting this equation from the equation for the conservation of the total energy, we find the **internal energy equation**:

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \vec{\nabla}\cdot(\rho\varepsilon\vec{u}) = -P\vec{\nabla}\cdot\vec{u} - \vec{\nabla}\cdot\vec{F} + \Psi \ , \tag{1.46}$$

where

$$\Psi \equiv \sum_{i,j} \pi_{ij} \frac{\partial u_i}{\partial x_j} \ . \tag{1.47}$$

The function Ψ is the **rate of viscous dissipation**; viscosity converts bulk kinetic energy into internal energy.

The partial time derivative of a function Q, which we have written as $\partial Q/\partial t$, is the rate of change as viewed by an observer at a fixed coordinate position, who is watching the gas flow by. Such an observer is called an **Eulerian** observer. An alternative point of view is that of an observer who is moving along with the bulk flow of the gas. Such an observer is called a **Lagrangian** observer.

The Lagrangian time derivative,

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + (\vec{u} \cdot \vec{\nabla})Q \ , \tag{1.48}$$

is the rate of change of Q as seen by a Lagrangian observer, who is moving along with a designated parcel of gas. [Note that $(\vec{u} \cdot \vec{\nabla})\vec{Q} = (\vec{u} \cdot \vec{\nabla} Q_x)\hat{\imath} + (\vec{u} \cdot \vec{\nabla} Q_y)\hat{\jmath} + (\vec{u} \cdot \vec{\nabla} Q_z)\hat{k}$).

In a Lagrangian form, the **continuity equation** is

$$\frac{D\rho}{Dt} = -\rho \vec{\nabla} \cdot \vec{u} \ . \tag{1.49}$$

The **momentum equation** is

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla} P + \frac{1}{\rho}\vec{\nabla} \cdot \overleftrightarrow{\pi} + \vec{g} \ . \tag{1.50}$$

The acceleration is provided by the pressure gradient, the viscous drag, and by the gravitational and electromagnetic forces that we have included in $\vec{g}$.

The **internal energy equation** is

$$\frac{D\varepsilon}{Dt} = -\frac{P}{\rho}\vec{\nabla} \cdot \vec{u} - \frac{1}{\rho}\vec{\nabla} \cdot \vec{F} + \frac{1}{\rho}\Psi \ . \tag{1.51}$$

The change in internal energy is given by a PdV work term, a heat conduction term, and a viscous heating term.

Equations (1.49), (1.50), and (1.51) are valid for *nonradiative* gas dynamics. Since this is a course on radiative gas dynamics, we must take into account the effect of radiation on the momentum and energy of the gas. To the right hand side of the internal energy equation, we must add the term $(\Gamma - \Lambda)/\rho$, where Γ is the volumetric heating rate, and Λ is the volumetric cooling rate.[5]

The equations for conservation of mass, momentum, and energy provide *five* equations in all. How many unknowns are there? There's the density ρ, the pressure P, the bulk velocity $\vec{u}$ (3 unknowns), the conduction heat flux $\vec{F}$ (3 unknowns), and the viscous stress tensor $\overleftrightarrow{\pi}$ (5 independent components). The energy, for a monatomic gas, is given by the relation $\varepsilon = 3P/(2\rho)$, the rate of viscous dissipation can be computed from $\overleftrightarrow{\pi}$ and $\vec{u}$, the acceleration due to self-gravity is given by Poisson's equation

$$\vec{\nabla} \cdot \vec{g} = -4\pi G\rho \ , \tag{1.52}$$

and the acceleration from any external sources is assumed to be known.

So, we have 5 equations and 13 unknowns. The clever way to break out of the hierarchy of equations is to express the viscosity $\overleftrightarrow{\pi}$ and the heat flux $\vec{F}$ in terms of ρ, P, and $\vec{u}$. This will leave us with 5 equations and 5 unknowns. Once we specify our initial conditions and boundary conditions, we can proceed to solve the equations.

[5] In certain cases, such as in stars that exceed the Eddington Limit, radiative pressure has an important effect on the motion of the gas, and must be included in the momentum conservation equation.