Chapter 8

Spherical Accretion

Accretion may be defined as the gravitational attraction of material onto a compact object. The compact object may be a black hole with a Schwarzschild radius $R_* = 2GM_*/c^2 \sim 3\,\mathrm{km}(M_*/\,\mathrm{M}_\odot)$. Another possible compact object is a neutron star, with mass $M_* \sim 2\,\mathrm{M}_\odot$ and radius $R_* \sim 10\,\mathrm{km}$. Yet another possible compact object is a white dwarf, with mass $M_* \sim 1\,\mathrm{M}_\odot$ and radius $R_* \sim 10^4\,\mathrm{km}$. The accretion of gas onto a compact object can be a very efficient way of converting gravitational potential energy into radiation. Consider a compact object of mass M_* and radius R_*. If a blob of hydrogen with mass m is allowed to drop onto the compact object, starting at infinity, the amount of energy released is

$$E_{\mathrm{acc}} = \frac{GM_* m}{R_*} = \frac{R_{\mathrm{Sch}}}{2R_*} mc^2 \ , \tag{8.1}$$

where R_{Sch} is the Schwarzschild radius of the mass M_*. By comparison, the conversion of the hydrogen into helium would yield an energy $E_{\mathrm{nuc}} = 0.007mc^2$. Thus, accretion accretion onto a compact object with $R_* \lesssim 70R_{\mathrm{Sch}}$, such as a black hole or neutron star, is a more efficient mechanism than nuclear fusion of H to He.

There is a limit to the rate $\dot{M}$ at which a compact object can accrete matter. Suppose the infalling matter consists of ionized hydrogen. The luminosity L of the central compact object exerts a radiation force on the free electrons by Thomson scattering. The outward radial force on an electron at a radius r is $f_{\mathrm{out}} = \sigma_T L/(4\pi r^2 c)$, where $\sigma_T = 6.7 \times 10^{-25}\,\mathrm{cm}^2$ is the Thomson cross section of the electron. As each electron moves outward, it drags a proton along with it in order to conserve charge neutrality. The inward gravitational force on the electron - proton pair is $f_{\rm in} = GM_* m_p/r^2$. There exists a limiting luminosity, called the **Eddington luminosity**, at which the two forces cancel:

$$L_{\rm Edd} = \frac{4\pi G M_* m_p c}{\sigma_T} = 3.4 \times 10^4\,{\rm L}_\odot \left(\frac{M_*}{{\rm M}_\odot}\right) . \tag{8.2}$$

When the luminosity of the central compact object is greater than this value, the surrounding hydrogen gas will be blown away by the radiation pressure. If the Eddington luminosity is emitted as black-body radiation, the temperature will be

$$T_{bb} = \left(\frac{L_{\rm Edd}}{4\pi R_*^2 \sigma_{SB}}\right)^{1/4} = \left(\frac{G M_* m_p c}{R_*^2 \sigma_{SB}}\right)^{1/4} . \tag{8.3}$$

where σ_{SB} is the Stefan-Boltzmann constant, and R_* is the radius of the surface from which the radiation is emitted (for a black hole, of course, this surface will be outside the Schwarzschild radius). For a black hole accreting at the Eddington limit, the temperature of the radiation will be $T_{bb} \sim 4 \times 10^7\,{\rm K}(M_*/\,{\rm M}_\odot)^{-1/4}$, if the radiation comes from immediately outside the Schwarzschild radius. The spectrum of the emitted photons will then peak at a photon energy $E \sim 20\,{\rm keV}(M_*/\,{\rm M}_\odot)^{-1/4}$.

The existence of the Eddington luminosity implies the existence of a maximum accretion rate, $\dot{M}_{\rm Edd}$, for an accreting compact object. If the accretion energy $E_{\rm acc}$ is converted entirely into radiation, then the luminosity is $L_{\rm acc} = GM_*\dot{M}/R_*$, and the maximum possible accretion rate is

$$\dot{M}_{\rm Edd} = \frac{4\pi m_p c R_*}{\sigma_T} = 9 \times 10^{16}\,{\rm g\,sec^{-1}}(\frac{R_*}{1\,{\rm km}}) = 1 \times 10^{-3}\,{\rm M}_\odot\,{\rm yr}^{-1}(\frac{R_*}{{\rm R}_\odot}) . \tag{8.4}$$

In reality, the conversion is not 100% efficient, the accretion is not perfectly spherically symmetrical, and the radiation is not perfectly spherically symmetrical; thus, matter can be accreted at rates somewhat greater than $\dot{M}_{\rm Edd}$.

Quasars and active galactic nuclei can have luminosities as great as $10^{14}\,{\rm L}_\odot$. If they are powered by accretion onto a central compact object, its mass must be $M_* \gtrsim 3 \times 10^9\,{\rm M}_\odot$, with a Schwarzschild radius $R_{\rm Sch} \gtrsim 8$ light-hours. The X-ray variability of bright AGNs and quasars is on the scale of hours, suggesting that the central accreting object must be a black hole near its Eddington luminosity; otherwise its radius and variability timescale would be too large. If the central object is a black hole, the rate at which it must accrete matter in order to radiate with $L = 10^{14}\,{\rm L}_\odot$ is $\dot{M} = L/(c^2\eta) \sim (\eta^{-1}) 7\,{\rm M}_\odot\,{\rm yr}^{-1}$, where

η is the efficiency with which the accretion energy is converted to radiation. The most luminous X-ray binaries within our own galaxy have luminosities of about $3\times10^4\,\mathrm{L}_\odot$, implying a mass of $1\,\mathrm{M}_\odot$ if they are powered by accretion at the Eddington limit.

Now, let us consider the dynamics of spherically symmetric accretion. A compact object of mass M is embedded in the ISM. Far from the accreting compact object, the medium has a uniform density ρ_∞ and a uniform pressure P_∞; the sound speed far from the accreting object thus has the value $a_\infty = (\gamma P_\infty/\rho_\infty)^{1/2}$. If the infall of matter has reached a steady state, then the equation of mass conservation reduces to the form $\dot{M} = -4\pi r^2 \rho u$, where $\dot{M}$ is the rate of accretion onto the compact object. Ignoring the self-gravity of the accreting gas, the equation of momentum conservation is

$$u\frac{du}{dr} + \frac{a^2}{\rho}\frac{d\rho}{dr} + \frac{GM}{r^2} = 0\ . \tag{8.5}$$

From the equation of mass conservation, we know that

$$\frac{1}{\rho}\frac{d\rho}{dr} = -\frac{2}{r} - \frac{1}{u}\frac{du}{dr}. \tag{8.6}$$

Substituting this result into the momentum conservation equation, we find the result

$$\frac{1}{2}\left(1 - \frac{a^2}{u^2}\right)\frac{d}{dr}(u^2) = -\frac{GM}{r^2}\left[1 - \frac{2a^2 r}{GM}\right]\ . \tag{8.7}$$

This equation is called the "Bondi equation", after Hermann Bondi, who pioneered the study of spherical accretion. It represents spherically symmetric, steady state accretion of a non-self-gravitating gas.[1]

Now consider the term in square brackets on the right hand side of the Bondi equation, above. As $r \to \infty$, the sound speed a approaches the finite value a_∞. Thus, the term in square brackets is negative at large radii. As $r \to 0$, however, the term in square brackets tends to increase. At some radius r_s, it will equal zero, unless some form of heating raises the sound speed to values $a(r)^2 > GM/(2r)$. Let us assume that there is a radius r_s at which $2a(r_s)^2 r_s = GM$, and the right hand side of the Bondi equation vanishes. At the radius r_s, the left hand side of the equation must also vanish. This means that either

$$u(r_s)^2 = a(r_s)^2 \tag{8.8}$$

[1]Bondi, 1952, MNRAS, 112, 195

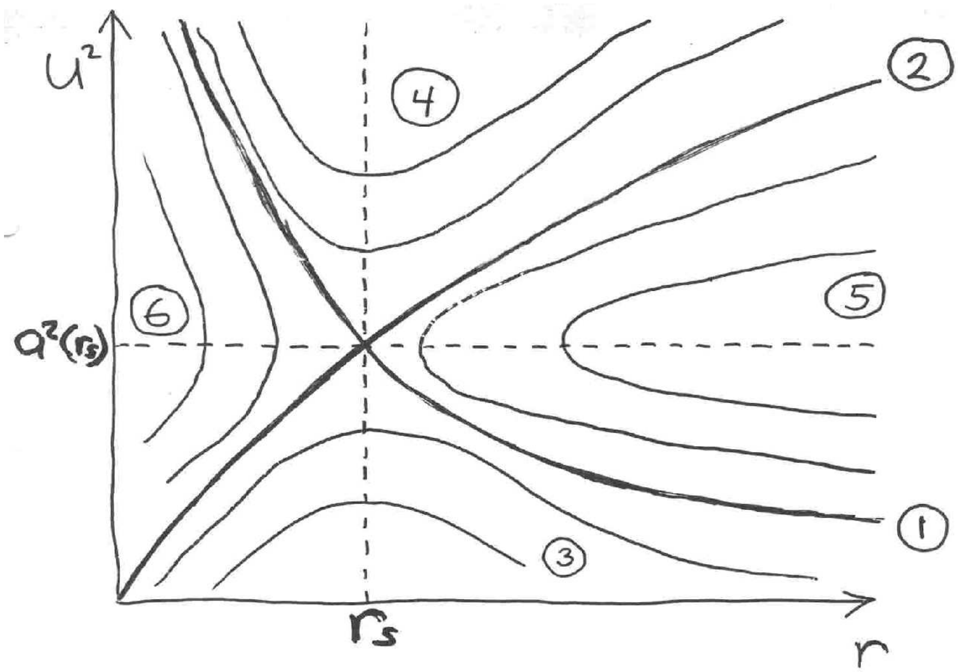

Figure 8.1: Solutions to the Bondi equation, divided into six families.

or

$$\frac{d(u^2)}{dr}\Big|_{r=r_s} = 0 \ . \tag{8.9}$$

There are six families of solutions to the Bondi equation, characterized by their behavior at the radius r_s and in the limits $r \to \infty$ and $r \to 0$. These families are illustrated in Figure 8.1.

$$\textit{Type 1:} \quad u(r_s)^2 = a(r_s)^2 \ , \quad u^2 \to 0 \text{ as } r \to \infty \ . \tag{8.10}$$

$$\textit{Type 2:} \quad u(r_s)^2 = a(r_s)^2 \ , \quad u^2 \to 0 \text{ as } r \to 0 \ . \tag{8.11}$$

The type 1 and type 2 solutions are known as the **transonic** solutions; in these solutions the radius r_s is known as the **sonic point**.

$$\textit{Type 3:} \quad \frac{d(u^2)}{dr}\Big|_{r=r_s} = 0 \ , \quad u^2 < a^2 \ . \tag{8.12}$$

$$\textit{Type 4:} \quad \frac{d(u^2)}{dr}\Big|_{r=r_s} = 0 \ , \quad u^2 > a^2 \ . \tag{8.13}$$

The type 3 and type 4 families of solution represent flow that is everywhere subsonic (type 3) or everywhere supersonic (type 4).

$$\textit{Type}\quad \textit{5:}\qquad u(r_s)^2 = a(r_s)^2\ ,\quad r > r_s\ . \tag{8.14}$$

$$\textit{Type}\quad \textit{6:}\qquad u(r_s)^2 = a(r_s)^2\ ,\quad r < r_s\ . \tag{8.15}$$

The type 5 and type 6 families of solutions represent solutions that are mathematically permissible but physically impossible. They are double-valued, giving two values of u^2 at a given value of r. Since the bulk velocity u must have a unique value at every point, we exclude families (5) and (6) as possible solutions for spherical accretion. Moreover, for the case of spherical accretion, we want solutions for which $u^2 \to 0$ as $r \to \infty$. We thus exclude family (4) and solution (2) from consideration, since they have supersonic infall at large radius. This leaves us with two possible solutions. The solutions of type (3) represent an accretion flow that is subsonic at all radii; they consist of a gentle infall that gradually settles into hydrostatic equilibrium at small radii. In most cases, accretion is not gentle; the accreted matter comes zooming in at supersonic speeds at small radii. Therefore, we will focus our attention on solution (1), which represents an accretion flow that is subsonic at $r > r_s$ and supersonic at $r < r_s$.

If the accretion is transonic, of type (1), then we can uniquely determine the accretion rate $\dot{M}_t$ in terms of the mass M of the accreting object and the density ρ_∞ and the sound speed a_∞ at infinity. Let us suppose that the infall is adiabatic, with $P \propto \rho^\gamma$. The equation of momentum conservation can then be integrated to yield the result

$$\frac{u(r)^2}{2} + \frac{a(r)^2}{\gamma - 1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma - 1}\ . \tag{8.16}$$

This quantity is known as the **Bernoulli integral**. Evaluating the Bernoulli integral at the sonic point r_s tells us that

$$a(r_s)^2 \left(\frac{1}{2} + \frac{1}{\gamma - 1} - 2 \right) = \frac{a_\infty^2}{\gamma - 1}\ , \tag{8.17}$$

or

$$a(r_s) = a_\infty \left(\frac{2}{5 - 3\gamma} \right)^{1/2}\ . \tag{8.18}$$

This implies that

$$r_s = \frac{5 - 3\gamma}{4} \frac{GM}{a_\infty^2} \tag{8.19}$$

and

$$\rho(r_s) = \rho_\infty \left(\frac{2}{5 - 3\gamma} \right)^{1/(\gamma-1)} . \tag{8.20}$$

When the adiabatic index is $\gamma = 5/3$, the accreting gas doesn't reach the sonic point until $r_s = 0$.

Using the relation $\dot{M} = 4\pi r_s^2 \rho(r_s) a(r_s)$, we find that the transonic accretion rate for the type (1) solution is

$$\dot{M}_t = 4\pi q_s \frac{G^2 M^2 \rho_\infty}{a_\infty^3} , \tag{8.21}$$

where

$$q_s(\gamma) = \frac{1}{4} \left(\frac{2}{5 - 3\gamma} \right)^{(5-3\gamma)/(2\gamma-2)} . \tag{8.22}$$

The numerical value of q_s ranges from $q_s = 1/4$ at $\gamma = 5/3$ to $q_s = e^{3/2}/4 \approx 1.12$ when $\gamma = 1$. If the accreting matter is ionized hydrogen, the transonic accretion rate has the value

$$\dot{M}_t = 1.2 \times 10^{10}\,\mathrm{g\,sec^{-1}} \left(\frac{M}{\mathrm{M_\odot}} \right)^2 \left(\frac{\rho_\infty}{10^{-24}\,\mathrm{g\,cm^{-3}}} \right) \left(\frac{T_\infty}{10^4\,\mathrm{K}} \right)^{-3/2} . \tag{8.23}$$

This amounts to about $10^{-16}\,\mathrm{M_\odot\,yr^{-1}}$ for a $1\,\mathrm{M_\odot}$ accreting mass. The Eddington accretion rate for a black hole is $\dot{M}_{\rm Edd} \sim 3 \times 10^{17}\,\mathrm{g\,sec^{-1}}(M/\,\mathrm{M_\odot})$, so an isolated black hole in the ISM is in little danger of exceeding its Eddington accretion rate.

The relation between the bulk velocity $u(r)$ and the sound speed $a(r)$ can be computed from the equation

$$-u = \frac{\dot{M}}{4\pi r^2 \rho(r)} = \frac{\dot{M}}{4\pi r^2 \rho_\infty} \left(\frac{a_\infty}{a(r)} \right)^{2/(\gamma-1)} . \tag{8.24}$$

Thus

$$u = -\frac{q_s G^2 M^2}{r^2 a_\infty^3} \left(\frac{a(r)}{a_\infty} \right)^{-2/(\gamma-1)} , \tag{8.25}$$

or

$$\frac{u}{a_\infty} = -\frac{q_s}{4}\left(\frac{r}{r_a}\right)^{-2}\left(\frac{a(r)}{a_\infty}\right)^{-2/(\gamma-1)} , \tag{8.26}$$

with $r_a \equiv 2GM/a_\infty^2$. The "accretion radius" r_a is the radius at which the density ρ and sound speed a start to significantly increase above their ambient values of ρ_∞ and a_∞. The relation between the sonic radius and the accretion radius is $r_s = [(5-3\gamma)/8]r_a$. At large radii ($r \gg r_a$), the infall velocity, sound speed, and density of the transonic flow are

$$u \approx -\frac{q_s a_\infty}{4}\left(\frac{r}{r_a}\right)^{-2}\left[1 - \frac{1}{2}\frac{r_a}{r}\right] \tag{8.27}$$

$$a \approx a_\infty\left[1 + \frac{\gamma-1}{4}\frac{r_a}{r}\right] \tag{8.28}$$

$$\rho \approx \rho_\infty\left[1 + \frac{1}{2}\frac{r_a}{r}\right] . \tag{8.29}$$

For a gas with $\gamma = 5/3$, the infall velocity, sound speed, and density at small radii ($r \ll r_a$) are

$$u \approx -a \approx -\frac{a_\infty}{2}\left(\frac{r}{r_a}\right)^{-1/2} \tag{8.30}$$

$$\rho \approx \frac{\rho_\infty}{8}\left(\frac{r}{r_a}\right)^{-3/2} . \tag{8.31}$$

If $1 \leq \gamma < 5/3$, the infall at $r \ll r_s$ is supersonic, and the infalling gas is in free fall. From the Bernoulli integral, we find that $u^2/2 \approx GM/r$, or

$$u \approx -a_\infty\left(\frac{r}{r_a}\right)^{-1/2} . \tag{8.32}$$

The density is then

$$\rho \approx \frac{q_s \rho_\infty}{4}\left(\frac{r}{r_a}\right)^{-3/2} . \tag{8.33}$$

Spherical accretion of gas thus has a characteristic density profile, with $\rho \propto r^{-3/2}$ at small radii and $\rho =$ constant at large radii.

The gas will come pouring in with velocity $u \propto r^{-1/2}$ at small radii, while the temperature of the gas increases to the value

$$T \approx \frac{T_\infty}{4}\left(\frac{r}{r_a}\right)^{-1} \tag{8.34}$$

when $\gamma = 5/3$ and

$$T \approx T_\infty \left(\frac{q_s}{4}\right)^{\gamma-1} \left(\frac{r}{r_a}\right)^{-3(\gamma-1)/2} \tag{8.35}$$

when $1 \leq \gamma < 5/3$. The velocity will not increase indefinitely; eventually the gas will slam into the surface of the compact object – or disappear within the event horizon if the compact object is a black hole. The velocity of the gas just before it hits the compact object will be $u \sim -(2GM/R) \sim -V_{\rm esc}$.

Spherical accretion is only a good approximation for isolated compact objects. If the accreting mass is fed by a binary companion, or is located in the center of an active galactic nucleus, an accretion disk will generally form. The case to which spherical accretion applies is the case of an isolated black hole or neutron star in the middle of the ISM. It is very unlikely, however, that a stellar remnant will be at rest with respect to the ambient ISM. Direct measurements of the radial velocities of pulsars indicate that pulsars in the disk of our galaxy have a velocity dispersion of $\sim 100\,{\rm km\,s^{-1}}$. This is much larger than the sound speed $a \sim 12\,{\rm km\,s^{-1}}$ in the warm neutral medium. Thus, a typical pulsar will be moving at supersonic speeds with respect to its ambient medium.

If the accreting body has a velocity V with respect to the ambient medium, the transonic accretion rate has the form

$$\dot{M}_t = 4\pi\tilde{q}\frac{G^2M^2\rho_\infty}{(a_\infty^2 + V^2)^{3/2}} \ , \tag{8.36}$$

where $\tilde{q}$ is a factor of order unity. When $V > a_\infty$, a bow shock forms in front of the accreting object, as shown in Figure 8.2, raising the temperature of the gas and decreasing its bulk velocity relative to the central accreting mass. At radii $r \ll r_a \sim 2GM/(V^2 + a_\infty^2)$, the flow of the gas is approximately radial, and takes the form of the spherically symmetric Bondi solution.

A compact object that is moving at highly supersonic speeds will accrete mass at a rate $\dot{M} \propto V^{-3}$. However, there is a lower limit to the accretion rate that is dictated by the fact that the compact object has a physical cross section equal to πR^2. Thus, in the limit that the accreting object is moving at a velocity V that is much greater than the escape velocity $V_{\rm esc} = (2GM/R)^{1/2}$ from its surface, the accretion rate is

$$\dot{M}_t \approx \pi R^2 \rho_\infty V \ . \tag{8.37}$$

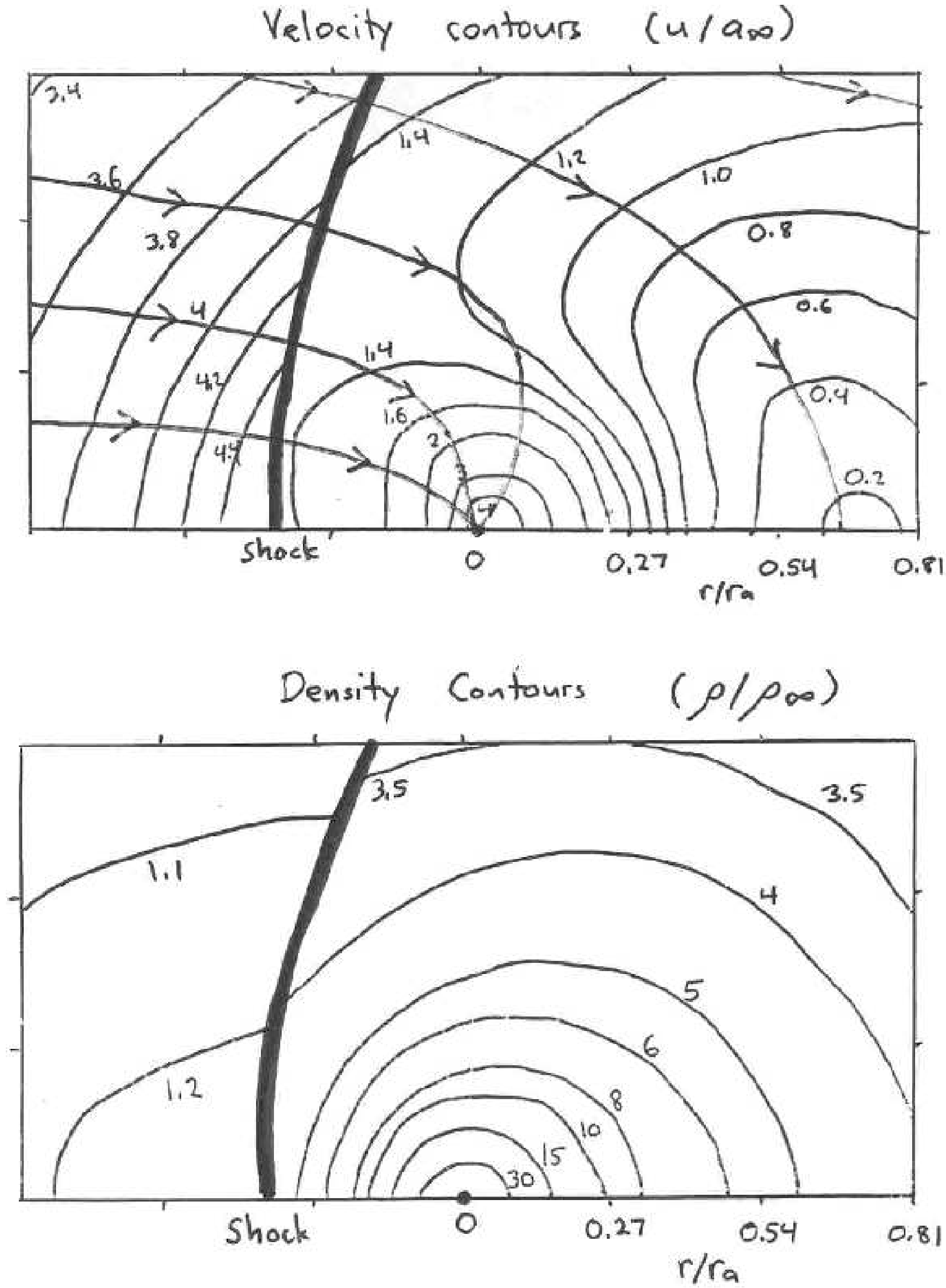

Figure 8.2: The isovelocity contours (above) and isodensity contours (below) for accretion onto a compact object moving with speed $V = 2.4a_\infty$ with respect to the ambient gas.

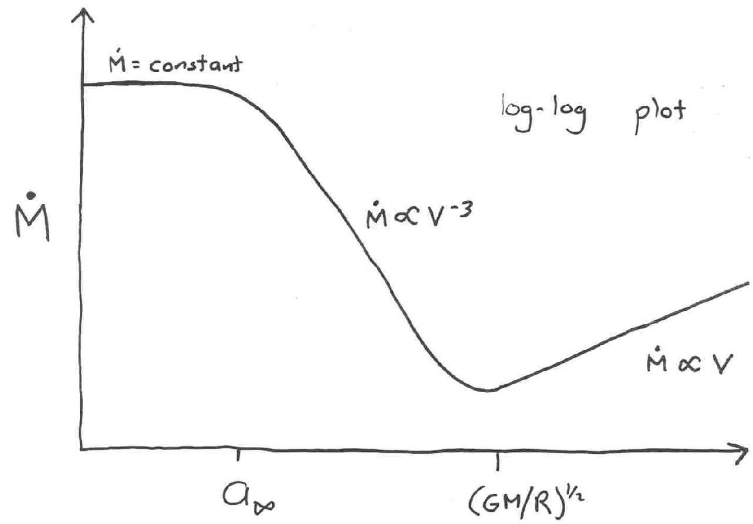

Figure 8.3: The mass accretion rate for a compact object as a function of the speed with which is it moving relative to the surrounding gas.

To minimize the rate of accretion for a compact object, you should send it off at a velocity equal to its escape velocity. A schematic plot of the accretion rate as a function of the compact object's velocity is given in Figure 8.3.