ASTRONOMY 294Z: The History of the Universe
Professor Barbara Ryden

SOLUTIONS TO PROBLEM SET # 3

1) [20 points] *You are given a ticket for running a red traffic light. For an observer halted at the light, the light emitted has a wavelength* $\lambda_0 = 700\,\mathrm{nm}$. *You tell the traffic cop that because you were approaching the light, the Doppler shift made it appear green* ($\lambda = 500\,\mathrm{nm}$)*. How fast would you have been going if this were true? If the speed limit is 35 miles per hour, and the speeding fine is $1 for every mile per hour over the limit, how large a fine would you have to pay?*

From the formula for the Doppler shift (lecture for Thursday, January 17), your speed relative to the light would be

$$v = \frac{\lambda - \lambda_0}{\lambda_0} c = \frac{500\,\mathrm{nm} - 700\,\mathrm{nm}}{700\,\mathrm{nm}} c = -\frac{2}{7} c \; . \qquad (1)$$

(The negative sign means that you are moving *toward* the light.) A speed of 2/7 the speed of light is no trifle. In kilometers per second, it would be

$$v = \frac{2}{7}(300{,}000\,\mathrm{km/sec}) = 85{,}714\,\mathrm{km/sec} \; . \qquad (2)$$

In miles per hour, it would be

$$v = 85{,}714\,\mathrm{km/sec}\left(\frac{0.621\,\mathrm{miles}}{1\,\mathrm{km}}\right)\left(\frac{3600\,\mathrm{sec}}{1\,\mathrm{hour}}\right) = 191{,}623{,}857\,\mathrm{mph} \; . \qquad (3)$$

The resulting fine would be

$$N_\$ = (191{,}623{,}857\,\mathrm{mph} - 35\,\mathrm{mph})\left(\frac{\$1}{1\,\mathrm{mph}}\right) = \$191{,}623{,}822 \; . \qquad (4)$$

Better to confess to running the red light!

2) [20 points] *Assume that a typical galaxy contains 100 billion stars, and that there is one galaxy per cubic megaparsec, on average. How many galaxies are within a Hubble distance,* $c/H_0 = 4300\,\mathrm{Mpc}$, *of us? How many stars are within a Hubble distance of us?*

The volume of space within a Hubble distance of us constitutes a sphere of radius $r = c/H_0$. Thus, its volume will be

$$V = \frac{4\pi}{3} r^3 = \frac{4\pi}{3}\left(\frac{c}{H_0}\right)^3 = \frac{4\pi}{3}(4300\,\mathrm{Mpc})^3 = 3.33 \times 10^{11}\,\mathrm{Mpc}^3 \ . \qquad (5)$$

With one galaxy per cubic megaparsec, the total number of galaxies will be

$$N_{\mathrm{gal}} = 3.33 \times 10^{11}\,\mathrm{Mpc}^3 \left(\frac{1\,\mathrm{galaxy}}{1\,\mathrm{Mpc}^3}\right) = 3.33 \times 10^{11}\,\mathrm{galaxies} \ . \qquad (6)$$

Thus, there are roughly 300 billion galaxies within a Hubble distance of us, which is larger than the number of stars within our own galaxy. The total number of stars within a Hubble distance of us will be

$$N_{\mathrm{star}} = 3.33 \times 10^{11}\,\mathrm{galaxies}\left(\frac{10^{11}\,\mathrm{stars}}{1\,\mathrm{galaxy}}\right) = 3.33 \times 10^{22}\,\mathrm{stars} \ . \qquad (7)$$

3) [20 points] *The Sahara has an area of* $A = 9{,}000{,}000\,\mathrm{km}^2$. *The average depth of sand in the Sahara is* $d = 10\,\mathrm{m}$. *What is the total volume of sand in the Sahara, expressed in cubic meters? If an average grain of sand has a volume of* $1\,\mathrm{mm}^3$, *how many grains of sand are in the Sahara? Is the number of grains of sand in the Sahara greater than or less than the number of stars within a Hubble distance of us?*

The area of the Sahara, expressed in square meters, is

$$A = 9 \times 10^6\,\mathrm{km}^2 \left(\frac{1000\,\mathrm{m}}{1\,\mathrm{km}}\right)^2 = 9 \times 10^{12}\,\mathrm{m}^2 \ . \tag{8}$$

The total volume of the sand is its area times its depth:

$$V = A \times d = (9 \times 10^{12}\,\mathrm{m}^2)(10\,\mathrm{m}) = 9 \times 10^{13}\,\mathrm{m}^3 \ . \tag{9}$$

The volume of a single grain of sand, expressed in cubic meters, is

$$V_{\mathrm{grain}} = 1\,\mathrm{mm}^3 \left(\frac{1\,\mathrm{m}}{1000\,\mathrm{mm}}\right)^3 = 10^{-9}\,\mathrm{m}^3 \ . \tag{10}$$

Thus, the total number of grains of sand is the total volume divided by the volume of a single grain:

$$N_{\mathrm{grain}} = \frac{V}{V_{\mathrm{grain}}} = \frac{9 \times 10^{13}\,\mathrm{m}^3}{10^{-9}\,\mathrm{m}^3} = 9 \times 10^{22} \ . \tag{11}$$

This is greater than the number of stars we computed in problem 2.

4) [20 points] *The mass of the Sun is* $M_{\rm sun} = 2 \times 10^{33}\,{\rm grams}$*. The mass of a hydrogen atom is* $M_{\rm H} = 1.7 \times 10^{-24}\,{\rm grams}$*. If the Sun consisted entirely of hydrogen atoms, how many atoms would it contain? Dividing this number of atoms by the volume of the Sun, show how many hydrogen atoms there would be, on average, per cubic meter of the Sun.*

The total number of hydrogen atoms in the Sun is the mass of the Sun divided by the mass of a single hydrogen atom:

$$N_{\rm H} = \frac{M_{\rm sun}}{M_{\rm H}} = \frac{2 \times 10^{33}\,{\rm grams}}{1.7 \times 10^{-24}\,{\rm grams}} = 1.176 \times 10^{57} \ . \qquad (12)$$

This is, of course, gargantuanly bigger than the number of grains of sand in the Sahara, or the number of stars within a Hubble distance. The radius of the Sun is

$$R_{\rm sun} = 7 \times 10^5\,{\rm km} \left(\frac{1000\,{\rm m}}{1\,{\rm km}}\right) = 7 \times 10^8\,{\rm m} \ . \qquad (13)$$

The volume of the Sun then is

$$V_{\rm sun} = \frac{4\pi}{3} R_{\rm sun}^3 = \frac{4\pi}{3}(7 \times 10^8\,{\rm m})^3 = 1.437 \times 10^{27}\,{\rm m}^3 \ . \qquad (14)$$

The number of hydrogen atoms per cubic meter of the Sun is then

$$\frac{N_{\rm H}}{V_{\rm Sun}} = \frac{1.176 \times 10^{57}}{1.437 \times 10^{27}\,{\rm m}^3} = 8.18 \times 10^{29}\,{\rm m}^{-3} \ . \qquad (15)$$

5) [20 points] *If every star within a Hubble distance of us were as massive as the Sun and were made entirely of hydrogen, how many hydrogen atoms would be within a Hubble distance of us? Dividing this number of atoms by the volume of space within a Hubble distance of us, show how many hydrogen atoms there would be, on average, per cubic meter of the visible universe.*

The number of stars within a Hubble distance of us is $N_{\rm star} = 3.33 \times 10^{22}$. If each star were a blob of hydrogen equal in mass to the Sun, the number of hydrogen atoms per star would be $N_{\rm H} = 1.176 \times 10^{57}$. Multiplying these two numbers together, we find that the total number of hydrogen atoms within a Hubble distance would be

$$N = N_{\rm stars} \times N_{\rm H} = (3.33 \times 10^{22})(1.176 \times 10^{57}) = 3.92 \times 10^{79} \ . \qquad (16)$$

Yes, that's a lot of hydrogen atoms. However, they are distributed over a large volume. The volume within a Hubble distance of us is (from problem 2),

$$V = 3.33 \times 10^{11}\,{\rm Mpc}^3 \left(\frac{3.1 \times 10^{19}\,{\rm km}}{1\,{\rm Mpc}}\right)^3 \left(\frac{1000\,{\rm m}}{1\,{\rm km}}\right)^3 = 9.9 \times 10^{78}\,{\rm m}^3 \ . \quad (17)$$

Now we have the fun of dividing one hyper-gargantuan number by another to find the average number of hydrogen atoms per cubic meter:

$$\frac{N}{V} = \frac{3.92 \times 10^{79}}{9.9 \times 10^{78}\,{\rm m}^3} = 4\,{\rm m}^{-3} \ . \qquad (18)$$