1 Monday, November 21: Inverse Compton Scattering

When I did the calculations for the scattering of photons from electrons, I chose (for the sake of simplicity) the inertial frame of reference in which the electron was initially at rest. However, as we look at the universe around us, we see electrons in motion. The electron velocities, in our frame of reference, range from the highly non-relativistic motions of free electrons in partially ionized warm gas to the highly relativistic electrons that emit synchrotron radiation.

When an electron is initially moving relative to a photon, we can redo the analysis of the electron-photon collision, with the usual assumptions of the conservation of momentum and energy.[1] If the electron has an arbitrary initial velocity $\vec{v}_i$, the calculations become a bit tedious, and require a few envelope backs to compute. However, the special case in which the initial velocities of photon and electron are in opposite directions requires only a single envelope back.[2] Suppose that the photon is initially moving in the positive x direction, and the electron is initially moving in the negative x direction with velocity $-v_i\hat{e}_x$. The photon is scattered by an angle θ from its initial direction, and the electron is scattered by an angle φ from its initial direction. The initial momentum of the system will be

$$\vec{p}_i = -\gamma_i m_e v_i \hat{e}_x + \frac{h\nu_i}{c}\hat{e}_x \tag{1}$$

and its initial energy will be

$$\epsilon_i = \gamma_i m_e c^2 + h\nu_i \ . \tag{2}$$

The final momentum of the system will be

$$\vec{p}_f = -\gamma_f m_e v_f(\cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y) + \frac{h\nu_f}{c}(\cos\theta\hat{e}_x + \sin\theta\hat{e}_y) \tag{3}$$

[1] Equivalently, we can take our calculations with an initially stationary electron, and transform them to another frame of reference in which the electron initially has a constant velocity.

[2] I'll refer to this case as a "head-on" collision, in analogy with a collision between cars going in opposite directions; a "head-on" collision, however, doesn't imply that the photon and electron are necessarily reflected straight back the way they came.

and its final energy will be

$$\epsilon_f = \gamma_f m_e c^2 + h\nu_f \ . \tag{4}$$

Aside from the addition of the electron's initial momentum, and an acknowledgment that the electron's initial energy is greater than the rest energy, this is the same as the problem we solved last week. I will therefore leave it as an "exercise for the reader" to demonstrate that the final energy, in its dimensionless form, $X_f \equiv h\nu_f/(m_e c^2)$, is

$$X_f = X_i \frac{\gamma_i(1+\beta_i)}{\gamma_i(1+\beta_i\cos\theta) + X_i(1-\cos\theta)} \ , \tag{5}$$

where θ is the angle through which the photon is scattered, $\beta_i = v_i/c$ is the initial speed of the oncoming electron, in units of the speed of light, and γ_i is the initial Lorentz factor for the electron.[3]

The fractional amount of energy lost or gained by a photon in a "head-on" collision with an electron is

$$\frac{X_f - X_i}{X_i} = \frac{(\gamma_i\beta_i - X_i)(1-\cos\theta)}{\gamma_i(1+\beta_i\cos\theta) + X_i(1-\cos\theta)} \ . \tag{6}$$

Thus, there is no energy transfer when $X_i = \gamma_i\beta_i$. If the electron is initially highly relativistic ($\gamma_i \gg 1$), photons with energy $h\nu_i < \gamma_i m_e c^2$ will gain energy by a head-on collision; more energetic photons will lose energy. If the electron is initially highly non-relativistic ($\beta_i \ll 1$), photons with momentum $h\nu_i/c < m_e v$ will gain energy through head-on collisions; higher momentum photons will lose energy.

The highest fractional change in energy, for a given X_i and γ_i, occurs when the photon is reflected back the way it came ($\cos\theta = -1$). If a low-frequency photon ($X_i \ll 1$) is reflected in this way by a high-energy electron ($\gamma_i \gg 1$), the photon's fractional gain in energy can be large:

$$\frac{X_f - X_i}{X_i} \approx \frac{2\gamma_i\beta_i}{\gamma_i(1-\beta_i)} \approx 4\gamma_i^2 \ . \tag{7}$$

In an encounter with an electron with a Lorentz factor $\gamma \sim 1000$, a low-frequency photon can increase its energy by a factor of more than a million.

[3] I note, with some relief, that in the limit $\beta_i = 0$, $\gamma_i = 1$, I recover the correct formula for the case in which the electron is initially at rest (see the previous lecture).

The case in which the photon gains energy from an encounter with an electron ($X_i < \gamma_i \beta_i$ for head-on collisions) is known as *inverse Compton scattering*. One example of inverse Compton scattering that I've already mentioned is the Sunyaev-Zel'dovich effect, in which Cosmic Microwave Background (CMB) photons gain energy as they scatter from free electrons in the intracluster gas of rich galaxy clusters. As a general case, consider a gas with temperature T. As long as $T \ll m_e c^2/k \sim 6 \times 10^9\,\mathrm{K}$, we can treat the free electrons in the gas non-relativistically. That is, we can write down the typical speed of a free electron in the gas as

$$v_i \approx \sqrt{kT/m_e} \approx 0.012c \left(\frac{T}{10^6\,\mathrm{K}}\right)^{1/2} . \tag{8}$$

The critical frequency ν_c at which a photon will neither gain nor lose energy by an encounter with a typical thermal electron is given by $h\nu_c/c \approx m_e v$, or

$$h\nu_c \approx c\sqrt{m_e kT} \approx m_e c^2 \left(\frac{kT}{m_e c^2}\right)^{1/2} \sim 10\,\mathrm{keV} \left(\frac{T}{10^6\,\mathrm{K}}\right)^{1/2} . \tag{9}$$

(I confess: this is a extraordinarily crude calculation, which assumes that every electron-photon encounter is a head-on collision, and that every electron has a speed equal to the thermal speed. It yields only a rough estimate of the "break-even" frequency ν_c.) The average energy of a CMB photon is $h\nu \approx 6.3 \times 10^{-4}\,\mathrm{eV}$, corresponding to an energy of $X = h\nu/(m_e c^2) \approx 1.2 \times 10^{-9}$. Thus, when CMB photons scatter from the $T \sim 10^7\,\mathrm{K}$ electrons inside a cluster, they gain energy from the encounter. A head-on encounter yields a final energy

$$X_f \approx X_i \frac{1 + \beta_i}{1 + \beta_i \cos\theta} \approx X_i[1 + \beta_i(1 - \cos\theta)] , \tag{10}$$

assuming $\beta_i \ll 1$ for the electrons in the cluster. Thus, we expect the Sunyaev-Zel'dovich effect to give fractional changes in the photon energy of order

$$\frac{X_f - X_i}{X_i} \approx \beta_i \approx \left(\frac{kT}{m_e c^2}\right)^{1/2} . \tag{11}$$

This will be a few percent for a cluster at a temperature of 10 million Kelvin. At these low photon energies ($X_i \sim 10^{-9}$), the cross-section for scattering is equal to the Thomson cross-section. As you've already calculated in Problem

Set #2, the probability of a CMB photon being scattered as it passes through the tenuous intracluster medium is only $P \sim 0.004$. Since the probability of scattering is small, and the energy shift for a scattered photon is small, the net change to the CMB energy density in passing through a cluster is $(\text{small})^2$. The effect of the Sunyaev-Zel'dovich effect on the initial Planck spectrum of the CMB (Figure 1) is to shift it slightly to higher frequencies.

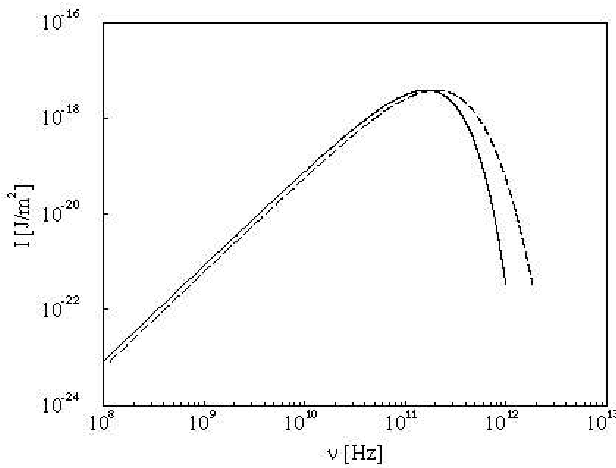

Figure 1: Solid: CMB spectrum without Sunyaev-Zel'dovich effect. Dashed: Spectrum with Sunyaev-Zel'dovich effect.

The intensity is decreased in the Rayleigh-Jeans portion of the spectrum (lowering the brightness temperature) and increased on the Wien tail (raising the brightness temperature).

Another situation where photons encounter hot gas is in the accretion disk associated with an active galactic nucleus (AGN). In the unified model of active galactic nuclei, the highly magnetized gas in the vicinity of the central black hole emits synchrotron radiation with a flux $F_\nu \propto \nu^{-0.7}$. Since the electrons are extremely energetic and the magnetic field is very strong in the central engine, you would expect the power-law synchrotron spectrum to extend to extremely high frequencies. When we observe the spectrum of a typical AGN, however (see Figure 2), we don't see a perfect power law at high frequencies. Instead, there is a broad "hump" centered at photon energies of $\sim 30\,\text{keV}$. This feature is sometimes called the "Compton hump". This name has been bestowed because it is thought to result from the scattering of synchrotron photons by the gas in the AGN accretion disk. If the scattering

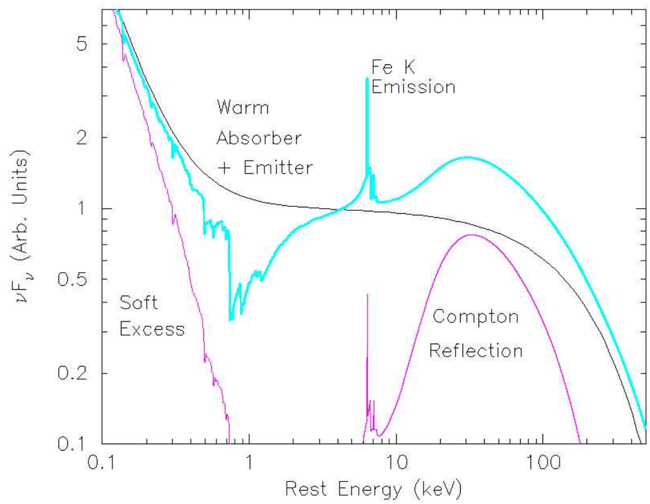

Figure 2: Average spectrum of an AGN (Seyfert 1, for the AGN connoisseurs in the audience), plotted as νF_ν versus $\log \nu$.

region of the AGN disk has a temperature T, then synchrotron photons with a frequency $\nu > \nu_c(T)$ will *lose* energy as they scatter from electrons in the disk. Synchrotron photons with $\nu < \nu_c(T)$ will *gain* energy as they scatter from electrons in the disk. Thus, there will be a pile-up of photons at energies $\sim h\nu_c$, resulting in the hump of the spectrum.

2 Wednesday, November 23: Inverse Compton Cooling

As an relativistic electron ($\epsilon_e = \gamma m_e c^2 \gg m_e c^2$) travels through a region containing low-energy photons ($\epsilon_\gamma = h\nu \ll m_e c^2$), the electron loses energy with each inverse Compton scattering. It is useful to know the rate $P_{\rm Compt}$, in ergs per second, at which an electron of Lorentz factor γ loses energy as it moves through a region of photon energy density $U_{\rm ph}$. In certain limiting cases, the computation is simple. Let us assume, to begin with, that the electron has $\gamma > 1$ in the observer's frame of reference. The electron has a head-on encounter with a photon that has a dimensionless energy/momentum/frequency $X \equiv h\nu/(m_e c^2)$ in the observer's frame. In the electron's rest frame, the energy of the photon will be blueshifted to $X' = X\gamma(1+\beta) \sim \gamma X$. If $X' \ll 1$, or equivalently, $X \ll 1/\gamma$, the electron-photon scattering will be a simple *Thomson scattering* in the electron's rest frame. Thomson scattering is very pleasant to work with, since it has a cross-section $\sigma_T = 6.65 \times 10^{-25}\,\mathrm{cm}^2$ that is independent of photon energy. Thus, for today, I'll not only assume that $X \ll 1$, which produces inverse Compton scattering, but I'll also make the more restrictive assumption that $X \ll 1/\gamma$, which reduces to simple Thomson scattering in the electron's initial rest frame. For visible light, which has $\nu_{\rm vis} \sim 5 \times 10^{14}\,\mathrm{Hz}$, the dimensionless photon energy is $X_{\rm vis} \sim 4 \times 10^{-6}$. Thus, electrons with $\gamma \ll 1/X_{\rm vis} \sim 2 \times 10^5$ will satisfy the criterion that $X_{\rm vis} \ll 1/\gamma$.

In the limit $X \ll 1/\gamma$, the computation of $P_{\rm Compt}$ is fairly straightforward. (I refer you to section 7.2 of the textbook for details.) The result, integrated over all collision angles, is

$$P_{\rm Compt} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_{\rm ph} \ . \tag{12}$$

This equation may be giving you a feeling of déja vu. Back on November 7, we computed the power radiated by an electron in the form of *synchrotron* radiation. When averaged over all possible pitch angles, the result was

$$P_{\rm synch} = \frac{4}{9} r_0^2 c \gamma^2 \beta^2 B^2 = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_B \ , \tag{13}$$

where $U_B = B^2/(8\pi)$ is the energy density of the magnetic field in which the electron is spiraling along.[4] Thus (as long as $X \ll 1/\gamma$) the ratio of an electron's inverse Compton scattering to its synchrotron emission is simple:

$$\frac{P_{\rm Compt}}{P_{\rm synch}} = \frac{U_{\rm ph}}{U_B} \ . \tag{14}$$

Suppose a high-energy electron is moving through the universe; you want to know whether it's losing energy primarily through inverse Compton scattering or synchrotron emission. All you need to do is compute the energy density of photons and of the magnetic field in its vicinity.

There exists a ubiquitous supply of low-energy photons in the universe: the Cosmic Microwave Background. The energy density of CMB photons is

$$U_{\rm CMB} = a T_{\rm CMB}^4 = 4.2 \times 10^{-13}\,\mathrm{erg\,cm^{-3}} (1+z)^4 \ , \tag{15}$$

[4]The similarity of the inverse Compton and synchrotron equations shouldn't be too astonishing; they both describe the interaction of an electron with an electromagnetic field.

where z is the cosmological redshift. Inverse Compton scattering from the CMB will be more effective than synchrotron emission when $U_{\rm CMB} > U_B$, or

$$B < \sqrt{8\pi a} T_{\rm CMB}^2 \approx 3.2 \times 10^{-6}\,{\rm gauss}(1+z)^2 \ . \tag{16}$$

High energy electrons will always lose energy. Even if you're in a region where the magnetic field is less than a microgauss, inverse Compton scattering from the CMB will always act to cool the electrons.

Suppose you have a power-law distribution of highly relativistic electrons ($\beta \approx 1$) with Lorentz factors ranging from $\gamma_{\rm min}$ to $\gamma_{\rm max} \gg \gamma_{\rm min}$. If the power-law index is $p > 1$, we may write the number density of electrons with Lorentz factor in the range γ to $\gamma + d\gamma$ as

$$N(\gamma)d\gamma = (p-1)n_e(\gamma/\gamma_{\rm min})^{-p}\frac{d\gamma}{\gamma_{\rm min}} \ , \tag{17}$$

where n_e is the total number density of relativistic electrons. Integrating the Compton power per electron over the entire range of electron energies, we find

$$\frac{dP}{dV} = \int_{\gamma_{\rm min}}^{\gamma_{\rm max}} P_{\rm Compt} N(\gamma) d\gamma \tag{18}$$

$$\approx \frac{4}{3}\sigma_T c U_{\rm ph} n_e \gamma_{\rm min}^2 \frac{p-1}{3-p}\left[(\gamma_{\rm max}/\gamma_{\rm min})^{3-p} - 1\right] \ . \tag{19}$$

If $p \approx 2.5$, typical for observed distributions of relativistic charged particles, then the Compton power per unit volume is

$$\left.\frac{dP}{dV}\right|_{\rm rel} \approx 4\sigma_T c U_{\rm ph} n_e \gamma_{\rm min}^2 \left(\frac{\gamma_{\rm max}}{\gamma_{\rm min}}\right)^{1/2} \ . \tag{20}$$

Since the total energy density of the electrons, in the case $p \approx 2.5$, is

$$U_e = m_e c^2 \int_{\gamma_{\rm min}}^{\gamma_{\rm max}} n(\gamma)\gamma d\gamma \approx 3 n_e \gamma_{\rm min} m_e c^2 \ , \tag{21}$$

the inverse Compton cooling time for the relativistic electrons is

$$t_{\rm Compt} \approx U_e / \frac{dP}{dV} \approx \frac{3}{4}\frac{m_e c}{\sigma_T \sqrt{\gamma_{\rm min}\gamma_{\rm max}}}\frac{1}{U_{\rm ph}} \ . \tag{22}$$

Scaled to the present energy density of the CMB, this is

$$t_{\rm Compt} \approx \frac{2300\,{\rm Gyr}}{\sqrt{\gamma_{\rm min}\gamma_{\rm max}}} \left(\frac{4.2 \times 10^{-13}\,{\rm erg\,cm^{-3}}}{U_{\rm ph}} \right) \ . \tag{23}$$

For inverse Compton cooling from the CMB to be significant today ($t_{\rm Compt} < H_0^{-1}$), the relativistic electrons must have a characteristic Lorentz factor $(\gamma_{\rm min}\gamma_{\rm max})^{1/2} > 160$.

We can compare the result for relativistic electrons to the Compton power per unit volume from a distribution of non-relativistic thermal electrons. In this case, we expect $\gamma \approx 1$, $\langle \beta^2 \rangle \approx 3kT/(m_e c^2)$, and

$$\left.\frac{dP}{dV}\right|_{\rm therm} \approx 4\sigma_T c U_{\rm ph} n_e \left(\frac{kT}{m_e c^2} \right) \ . \tag{24}$$

Since the energy density of the nonrelativistic thermal electrons is $U_e = n_e(3kT/2)$, the inverse Compton cooling time for the electrons is

$$t_{\rm Compt} = U_e (dP/dV)^{-1} = \frac{3}{8} \frac{m_e c}{\sigma_T} \frac{1}{U_{\rm ph}} \ , \tag{25}$$

independent of the electron temperature T (as long as the electrons are non-relativistic, and the typical photon energy is much less than kT). Scaled to the energy density of the CMB, the inverse Compton cooling time is

$$t_{\rm Compt} = 1200\,{\rm Gyr} \left(\frac{4.2 \times 10^{-13}\,{\rm erg\,cm^{-3}}}{U_{\rm ph}} \right) \ . \tag{26}$$

Thus, inverse Compton cooling of non-relativistic electrons by the CMB is negligible today. However, the cosmological fans among you may want to show that the inverse Compton cooling time from the CMB was equal to the age of the universe at a redshift

$$1 + z = \left(\frac{3}{2} \sqrt{\Omega_{\rm matter,0}} H_0 t_{\rm Compt,0} \right)^{2/5} \approx 5.5 \ , \tag{27}$$

assuming that Ω in matter today is $\Omega_{\rm matter,0} = 0.3$. A redshift of $z \approx 4.5$ corresponds to an age for the universe of $t \approx 1.3\,{\rm Gyr}$. For the first gigayear or so, inverse Compton scattering was an important cooling mechanism for hot gas.

So far, in looking at the interaction of high-energy electrons with low-energy photons, I have focused on the cooling of the electrons. However, the loss of energy by electrons is necessarily accompanied by a gain in energy by the photons which scatter from them. Suppose, for instance, that photons with a blackbody spectrum, characterized by a temperature $T_{\rm rad}$, interact with highly relativistic electrons, with the usual power-law energy spectrum $N(\gamma) \propto \gamma^{-p}$. The typical dimensionless photon energy, prior to scattering, will be $X_i \sim kT_{\rm rad}/(m_e c^2)$. If $X_i \ll 1/\gamma_{\rm max}$, then each electron-photon scattering will be a Thomson scattering in the electron's rest frame, boosting the energy of the photon from X_i to $X_f \sim \gamma^2 X_i$. With a power-law distribution of γ, we expect a power-law distribution of scattered photons. It can be shown that the power per unit volume per unit energy of scattered photons will be

$$\frac{dP}{dV dX_f} \propto n_e (kT_{\rm rad})^{(p+5)/2} X_f^{-(p-1)/2} \ . \tag{28}$$

Since the dimensionless energy X_f is proportional to the frequency ν of the photon, this implies a power per unit volume per unit frequency for scattered photons of

$$\frac{dP}{dV d\nu} \propto n_e (kT_{\rm rad})^{(p+5)/2} \nu^{-(p-1)/2} \ . \tag{29}$$

Once again, we see the similarity between inverse Compton scattering and synchrotron radiation. The photons scattered by relativistic electrons have the same power-law spectrum as the photons produced by synchrotron radiation. Even the power-law index, $s = (p-1)/2$, is the same. (Notice also that the shape of the spectrum in equation (29) is dictated by the distribution of electron energies, not by the shape of the blackbody spectrum. If the scattered photons had been initially monochromatic, the spectrum of scattered light would have been the same.)