1 Monday, October 10: Potentials

I've already remarked on the similarity between the electrostatic force between two point charges,

$$F = \frac{q_1 q_2}{r^2} \ , \tag{1}$$

and the gravitational force between two point masses,

$$F = -G\frac{m_1 m_2}{r^2} \ . \tag{2}$$

The similarities between the two suggest that we can profitably describe electrostatic phenomena using some techniques stolen (excuse me, I mean "borrowed") from Newtonian gravitational theory.

For instance, in dealing with gravity, it is useful to use the scalar field known as the gravitational potential. The gravitational potential $\Phi_{\rm grav}(\vec{r})$ is defined such that the force $\vec{F}_{\rm grav}$ acting on a unit mass is

$$\vec{F}_{\rm grav} = -\vec{\nabla}\Phi_{\rm grav} \ . \tag{3}$$

For an arbitrary mass density $\rho(\vec{r})$, the gravitational potential is

$$\Phi_{\rm grav}(\vec{r}) = -G\int \frac{\rho(\vec{r}\,')}{|\vec{r}\,' - \vec{r}|} d^3 r' \ . \tag{4}$$

Thus, you find the gravitational potential by smoothing the mass density field with a very broad $(1/r)$ smoothing function. Minima in the potential correspond to dense regions in the universe.

The gravitational potential itself cannot be directly measured. We can detect its gradient at a given point by determining the force on a unit mass. We can detect the difference in potential between two points by determining the energy needed to lift a particle from the point of lower potential to the point of higher potential. However, you can always add an arbitrary constant to the gravitational potential,

$$\Phi_{\rm grav} \rightarrow \Phi_{\rm grav} + C \ , \tag{5}$$

without changing any of the physics. For a mass distribution of finite extent, it is conventional to choose C so that $\Phi_{\rm grav} \rightarrow 0$ as $\vec{r} \rightarrow \infty$; however, this is just a convention. The concept of the "potential" has proved to be very useful for Newtonian gravity (largely because it's easier to deal with the scalar field

$\Phi_{\rm grav}$ than with the vector field $\vec{F}$). It would be doubly useful if we could also express electromagnetic phenomena in terms of potentials.

Alas, there are a few complicating factors. It is true that the electrostatic force between two point charges is directly analogous to the gravitational force between two point masses. However, if the point charges are moving relative to each other, there is a magnetic force between them that has no analogy in Newtonian gravity. Thus, we need an extension of potential theory that will allow us to deal with magnetism. There's another complicating factor, as well. Newtonian gravity only deals with slowly moving particles ($v \ll c$). In studying electromagnetism, we must be prepared to deal with highly relativistic charged particles.

With these caveats in mind, let's see how we can define potentials for the electric field strength $\vec{E}(\vec{r}, t)$ and the magnetic flux density $\vec{B}(\vec{r}, t)$. Let's start with the simplest of Maxwell's equations:

$$\vec{\nabla} \cdot \vec{B} = 0 \ . \tag{6}$$

Because of the vector identity

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{f}) = 0 \tag{7}$$

for an arbitrary vector function $\vec{f}$, we see that the divergenceless magnetic flux density can be written in the form

$$\vec{B}(\vec{r}, t) = \vec{\nabla} \times \vec{A}(\vec{r}, t) \ , \tag{8}$$

where $\vec{A}(\vec{r}, t)$ is the *electromagnetic vector potential.*[1]

If we write $\vec{B}$ as the curl of $\vec{A}$, we can write another of Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \ , \tag{9}$$

as

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) \tag{10}$$

or

$$\vec{\nabla} \times \left[\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right] = 0 \ . \tag{11}$$

[1]Note how we are expanding our concept of "potential"; the gravitational force is the gradient of a scalar potential, but the magnetic flux density is the curl of a vector potential.

We can now use another vector identity,

$$\vec{\nabla} \times (\vec{\nabla} g) = 0 \ , \tag{12}$$

where g is an arbitrary scalar function. Comparing with equation (11), we see that we can write

$$\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\Phi \ , \tag{13}$$

where $\Phi(\vec{r}, t)$ is the *electromagnetic scalar potential.* In contrast to the gravitational force, which can be written purely as the gradient of a scalar potential, the electric field strength must be written as

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \ . \tag{14}$$

If we knew the vector potential $\vec{A}(\vec{r}, t)$ and the scalar potential $\Phi(\vec{r}, t)$, we could compute $\vec{E}$ and $\vec{B}$. But how do we find out the potentials?

When in doubt, go back to Maxwell's equations. Consider the equation

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho_q \ , \tag{15}$$

where ρ_q is the charge density and $\vec{D} = \epsilon\vec{E}$. If we assume that the dielectric constant ϵ is equal to one within the region of interest, we can write

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho_q \ , \tag{16}$$

or, in terms of the electromagnetic potentials,

$$-\nabla^2\Phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = 4\pi\rho_q \ . \tag{17}$$

This equation relates the potentials Φ and $\vec{A}$ to the charge density ρ_q. As such, it is the electromagnetic equivalent of Poisson's equation,

$$\nabla^2\Phi_{\rm grav} = 4\pi\rho \ , \tag{18}$$

which relates the gravitational potential $\Phi_{\rm grav}$ to the mass density ρ. Poisson's equation tells you that if you know ρ, you can compute the gravitational potential Φ (give or take a constant C). However, because of the presence of a vector potential $\vec{A}$ in electromagnetism, equation (17) does *not* enable you to compute Φ and $\vec{A}$, even if the charge density ρ_q is known exactly. Equation (17) is a single equation in four unknowns, Φ and the three orthogonal components of $\vec{A}$. If we are to solve for Φ and $\vec{A}$, more information is needed.

Fortunately, we still have one last Maxwell's equation that we haven't yet used:

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \ , \tag{19}$$

where $\vec{H} = \vec{B}/\mu$, with μ being the magnetic permeability. If we assume that μ as well as ϵ is effectively equal to one, the equation becomes

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \ . \tag{20}$$

In terms of the potentials $\vec{A}$ and Φ, this becomes

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) + \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\Phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = \frac{4\pi}{c}\vec{j} \ . \tag{21}$$

Dipping once more into our Big Box of Vector Identities, we find

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{f}) = -\nabla^2 \vec{f} + \vec{\nabla}(\vec{\nabla} \cdot \vec{f}) \ . \tag{22}$$

Thus, we can rewrite equation (21) as

$$[-\nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})] + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\Phi + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{j} \ . \tag{23}$$

A little manipulation yields the form given in the textbook:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left[\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right] = -\frac{4\pi}{c}\vec{j} \ . \tag{24}$$

The above equation relates the potentials Φ and $\vec{A}$ to the current $\vec{j}$. Since it constitutes three equations in one (one equation for each of the three orthogonal components of $\vec{j}$), taken in conjunction with equation (17), it constitutes a complete set of equations, which can be solved for Φ and $\vec{A}$, given ρ_q and $\vec{j}$.

Equation (24) is a fairly complicated equation; it has too many terms on the left-hand side to qualify as an "elegant" equation. It's a pity we can't simplify it...

Or can we?

Remember that the gravitational force $\vec{F}_{\rm grav} = -\vec{\nabla}\Phi_{\rm grav}$ is unchanged by the transformation

$$\Phi_{\rm grav} \to \Phi_{\rm grav} + C \ , \tag{25}$$

where C is an arbitrary constant. We have the freedom to choose a value of C that makes the math easier for us. Similarly, the magnetic flux density $\vec{B} = \vec{\nabla} \times \vec{A}$ is unchanged by the transformation

$$\vec{A} \to \vec{A} + \vec{\nabla}\Psi \ , \tag{26}$$

where Ψ is an arbitrary scalar function. If we make this substitution, however, we do make a change in $\vec{E}$:

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \qquad (27) \\ &\to -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\left(\frac{1}{c}\frac{\partial\psi}{\partial t}\right) \ . \end{aligned}$$

Thus, to maintain the electric field strength unchanged, the transformation

$$\vec{A} \to \vec{A} + \vec{\nabla}\Psi \tag{28}$$

must be accompanied by the transformation

$$\Phi \to \Phi - \frac{1}{c}\frac{\partial\Psi}{\partial t} \ . \tag{29}$$

The fancy term for fiddling with the potentials in this way is a *gauge transformation*. Just as we are free to add an arbitrary constant C to the gravitational potential $\Phi_{\rm grav}$, we are free to add the gradient of an arbitrary function Psi to $\vec{A}$, as long as we subtract $1/c$ times the time derivative of Ψ from Φ. We are at liberty to do this for any function $\Psi(\vec{r}, t)$; unless we are totally insane, however, we carefully choose a function Ψ that will make the math easier for us.

It is asserted in the textbook that there exists a function Ψ for any $\vec{A}$ and Φ such that the resulting gauge transformation produces

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0 \ . \tag{30}$$

This choice of Ψ is called the *Lorentz gauge*. It is a particularly useful gauge transformation since it transforms equations (17) and (24) into the more elegant forms

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho_q \tag{31}$$

and

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j} \ . \tag{32}$$

2 Wednesday, October 12: Charged Particles

If we adopt the Lorentz gauge (that is, if we choose Ψ in such a way that equation (30) holds true), the equations relating the electromagnetic potentials to the charge density and current can be written in the form

$$\Box^2\Phi = -4\pi\rho_q \tag{33}$$

$$\Box^2\vec{A} = -\frac{4\pi}{c}\vec{j} \ , \tag{34}$$

where $\Box^2$ is the d'Alembertian operator,

$$\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \ . \tag{35}$$

If we start, for simplicity, by considering a system with constant charge density $\rho_q(\vec{r})$ and current $\vec{j}(\vec{r})$, we find that the inhomogeneous wave equations for Φ and $\vec{A}$ reduce to

$$\nabla^2\Phi = -4\pi\rho_q \tag{36}$$

and

$$\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{j} \ . \tag{37}$$

These are exactly analogous to Poisson's equation, and thus have the solutions

$$\Phi = \int \frac{\rho(\vec{r}')d^3r'}{|\vec{r}' - \vec{r}|} \tag{38}$$

and

$$\vec{A} = \frac{1}{c}\int \frac{\vec{j}(\vec{r}')d^3r'}{|\vec{r}' - \vec{r}|} \ . \tag{39}$$

The magnetic flux density can be computed as $\vec{B} = \vec{\nabla} \times \vec{A}$, and, given the lack of time dependence, the electric field strength can be computed as $\vec{E} = -\vec{\nabla}\Phi$. It is useful to be able to borrow the techniques of Newtonian gravity in this way, but we also need to cope with the problems present by time-varying charge densities and currents.

Let's begin with the simplest time-varying system I can think of. A point charge is stationary at the origin of our coordinate system. Its charge $q(t)$ varies with time. Since the charge doesn't move, there is no current: $\vec{j} = 0$. The charge density is nonzero only at the origin: $\rho_q(\vec{r}, t) = q(t)\delta(\vec{r})$, where δ is the Dirac delta function in three dimensions. For this spherically symmetric system, the wave equation for Φ reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi q(t)\delta(\vec{r}) \ . \tag{40}$$

Everywhere except the origin, the right hand side of equation (40) vanishes. The general solution, when the right hand side is zero, is

$$\Phi(r \neq 0, t) = \frac{1}{r}[f_{\rm ret}(t - r/c) + f_{\rm adv}(t + r/c)] \ . \tag{41}$$

You can verify by substitution that this is a solution of equation (40) for any functions $f_{\rm ret}$ and $f_{\rm adv}$. The "retarded" function $f_{\rm ret}$ represents *outgoing* spherical waves; that is, the value of $f_{\rm ret}$ at the origin now ($t = 0$) is equal to the value at r at a *later* time $t = r/c$. The "advanced" function $f_{\rm adv}$, by contrast, represents *incoming* spherical waves; that is, the value of $f_{\rm adv}$ at the origin now ($t = 0$) is equal to the value at r at an *earlier* time $t = -r/c$. If the only waves in Φ are being caused by the variable charge at the origin, we expect the waves to be traveling outward from their source.[2] Thus, in the real universe, we expect $f_{\rm adv} = 0$. The value of $f_{\rm ret}$ is determined by the boundary condition at the origin. Suppose that the charge varies with a characteristic timescale $t_{\rm vary}$. Then as you approach the point charge, when you are at a distance $r \ll ct_{\rm vary}$, the radial derivatives of Φ become large compared to the time derivatives. In this limit, equation (40) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = -4\pi q(t)\delta(\vec{r}) \ . \tag{42}$$

[2] Otherwise, we would have a violation of causality, which is the sort of thing that makes classical physicists have nervous breakdowns.

This is the same as Poisson's equation for a single point mass, and has the solution

$$\Phi(r \ll ct_{\rm vary}, t) = \frac{q(t)}{r} \ . \tag{43}$$

By comparison with equation (41), we find that $f_{\rm ret}(t - r/c)$ must equal $q(t - r/c)$ for the boundary condition to hold true as $r \to 0$. Thus, the desired solution for the scalar potential is

$$\Phi(r, t) = \frac{q(t - r/c)}{r} \ . \tag{44}$$

Thus, if you are at a distance r from a point charge, the electric field you experience now, at time t, is determined by the charge as it was in the past, at time $t - r/c$.[3]

In general, if you are at a point $\vec{r}$ at a time t, your knowledge of point $\vec{r}'$ dates from the time $t - |\vec{r} - \vec{r}'|/c$. The time $\tau = t - |\vec{r} - \vec{r}'|/c$ is known as the *retarded time*. In general, for a time-dependent charge density $\rho_q(\vec{r}, t)$ and current $\vec{j}(\vec{r}, t)$, the scalar potential is

$$\Phi(\vec{r}, t) = \int \frac{\rho_q(\vec{r}', \tau) d^3r'}{|\vec{r} - \vec{r}'|} \ , \tag{45}$$

where τ is the retarded time. The vector potential is

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int \frac{\vec{j}(\vec{r}', \tau) d^3r'}{|\vec{r} - \vec{r}'|} \ . \tag{46}$$

These equations can be solved numerically for arbitrary charge densities and currents, but it is most enlightening, physically speaking, to look at the simplest possible cases.

Consider a single charged particle (an electron, for instance) that has electric charge q. It is moving along a trajectory $\vec{r}_q(t)$: this may be a straight line, or a circle, or a complex wiggle. The velocity of the charged particle is $\vec{v}_q(t) = \dot{\vec{r}}_q$. The charge density is then

$$\rho_q(\vec{r}, t) = q\delta(\vec{r} - \vec{r}_q(t)) \tag{47}$$

and the current is

$$\vec{j}(\vec{r}, t) = q\vec{v}_q(t)\delta(\vec{r} - \vec{r}_q) \ . \tag{48}$$

[3]Since knowledge about the changing charge is transported at the speed of light, it takes a time $t_{\rm travel} = r/c$ for that knowledge to reach you.

Suppose you are located at a position $\vec{r}$ at a time t, so that your present separation from the charged particle is $\vec{R}(t) = \vec{r} - \vec{r}_q(t)$. What will be the scalar and vector potentials at your space-time location? First of all, the values of $\Phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$ will depend not on the current distance $\vec{R}(t)$ to the particle, but on its distance at the retarded time τ, defined such that $R(\tau) = |\vec{r} - \vec{r}_q(\tau)| = c(t - \tau)$. A detailed calculation, laid out in section 3.1 of the textbook reveals that for a moving charged particle

$$\Phi(\vec{r}, t) = \frac{q}{\kappa(\tau) R(\tau)} \ , \tag{49}$$

where

$$\kappa(\tau) \equiv 1 - \frac{\vec{R}(\tau)}{R(\tau)} \cdot \frac{\vec{v}_q(\tau)}{c} \ . \tag{50}$$

Note that the extra factor κ equals when the particle was moving perpendicular to your line of sight at the retarded time τ. However, it can be significantly different from one if the charged particle was moving toward you or away from you at relativistic speeds. By a similar calculation, the vector potential is

$$\vec{A}(\vec{r}, t) = \frac{q \vec{v}_q(\tau)}{c \kappa(\tau) R(\tau)} \ . \tag{51}$$

The potentials for a point charge with arbitrary velocity (equations 49 and 51) are known as the *Liénard-Wiechert potentials*.[4] Problem set #3 gives you the opportunity to compute the L-W potentials for different types of motion.

3 Friday, October 14: Radiation from Charged Particles

Differentiating the Liénard-Wiechert potentials for a moving point charge is, as the textbook says, "straightforward but tedious". If you want to know the details, look in Jackson's Classical Electrodynamics, which specializes in straightforward tediousness.[5] The resulting $\vec{E} = -\vec{\nabla}\Phi + (1/c)(\partial \vec{A}/\partial t)$ and

[4] Please note the spelling of Professor Emil Wiechert's name; the textbook has it wrong. A quick Google, however, reveals that the misspelling Liénard-Wiechart is surprisingly common.

[5] Sometimes, to be fair, it deviates into devious tediousness.

$\vec{B} = \vec{\nabla} \times \vec{A}$ can be written in fairly compact form if we introduce the notation

$$\vec{\beta} \equiv \vec{v}_q / c \tag{52}$$

and

$$\hat{n} \equiv \vec{R}/R \ . \tag{53}$$

That is, $\vec{\beta}$ is the particle velocity in units of c and $\hat{n}$ is a unit vector pointing from the charged particle to the point $\vec{r}$ where the fields $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ are being measured. The electric field strength can be divided into two components:

$$\vec{E}(\vec{r}, t) = \vec{E}_{\rm vel} + \vec{E}_{\rm rad} \tag{54}$$

where the *velocity field* is

$$\vec{E}_{\rm vel} = q \left[\frac{(\hat{n} - \vec{\beta})(1 - \beta^2)}{\kappa^3 R^2} \right]_\tau \tag{55}$$

and the *radiation field*, otherwise known as the *acceleration field*, is

$$\vec{E}_{\rm rad} = \frac{q}{c} \left[\frac{\hat{n}}{\kappa^3 R} \times \left([\hat{n} - \vec{\beta}] \times \dot{\vec{\beta}} \right) \right]_\tau \ . \tag{56}$$

(The little τ subscript is to remind us that all quantities in the square brackets should be computed at the appropriate retarded time $\tau < t$.) The velocity field (equation 55) is proportional to R^{-2}; in the limit $\beta \to 0$, it reduces to $\vec{E}_{\rm vel} = \hat{n} q / R^2$. The radiation field (equation 56) falls off more slowly with radius: as R^{-1}. Unlike the velocity field, which is parallel to $\hat{n}$ in the limit $\beta \ll 1$, the radiation field is always perpendicular to $\hat{n}$.

The magnetic flux density for the Liénard-Wiechert potentials is always

$$\vec{B}(\vec{r}, t) = \left[\hat{n} \times \vec{E} \right]_\tau \ . \tag{57}$$

Thus, the total magnetic flux density can be divided into a velocity field $\vec{B}_{\rm vel} = \hat{n} \times \vec{E}_{\rm vel}$ and a radiation field $\vec{B}_{\rm rad} = \hat{n} \times \vec{E}_{\rm rad}$. Note that $\hat{n}$ is perpendicular to $\vec{E}_{\rm rad}$, and $\vec{B}_{\rm rad}$ is perpendicular to both, with $E_{\rm rad} = B_{\rm rad}$. These are just the properties we found for plane wave solutions to Maxwell's equations.

Equations (55) and (56) are relativistically correct, and can be used even in the limit $\beta \to 1$. However, to keep things simple, let's start by looking at the non-relativistic case, where $\beta \ll 1$. In the limit $\beta \to 0$, the components of the electric field reduce to

$$\vec{E}_{\rm vel}(\beta \ll 1) \approx q \left[\frac{\hat{n}}{R^2} \right]_\tau \tag{58}$$

and

$$\vec{E}_{\rm rad}(\beta \ll 1) \approx \frac{q}{c} \left[\frac{\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})}{R} \right]_\tau \approx \frac{q}{c^2} \left[\frac{\hat{n} \times (\hat{n} \times \vec{a}_q)}{R} \right]_\tau \ , \tag{59}$$

where $\vec{a}_q$ is the particle's acceleration. In the limit $\beta \ll 1$, the retarded time is $\tau \approx t - R(t)/c$. The related magnetic flux densities are

$$\vec{B}_{\rm vel}(\beta \ll 1) = 0 \tag{60}$$

and

$$\vec{B}_{\rm rad}(\beta \ll 1) = \hat{n} \times \hat{E}_{\rm rad}(\beta \ll 1) \ . \tag{61}$$

Thus, the "velocity field", in this limit, represents an electrostatic field with $E_{\rm vel} \propto 1/R^2$, while the "radiation field" $\vec{E}_{\rm rad}$, taken together with $\vec{B}_{\rm rad}$ represents an electromagnetic wave generated by an accelerated particle ($a_q \neq 0$).

Note that, in the non-relativistic limit, the ratio of the radiation electric field to the velocity electric field will be of order

$$\frac{E_{\rm rad}}{E_{\rm vel}} \sim \frac{a_q R}{c^2} \ . \tag{62}$$

Thus, as long as the acceleration of the charged particle is non-zero (and as long as it isn't directed entirely along the line of sight), the radiation field will dominate over the velocity field at large radii, $R \gg c^2/a_q$. The region where the radiation field dominates is called the "far zone" or the "far-field region".

Suppose, for the moment, we are in the far zone of a non-relativistic charged particle. The electric field strength is thus dominated by the radiation field

$$\vec{E} = \frac{q}{c^2} \left[\frac{\hat{n} \times (\hat{n} \times \vec{a}_q)}{R} \right]_\tau \ , \tag{63}$$

with a corresponding magnetic flux density

$$\vec{B} = \left[\hat{n} \times \vec{E} \right]_\tau \ . \tag{64}$$

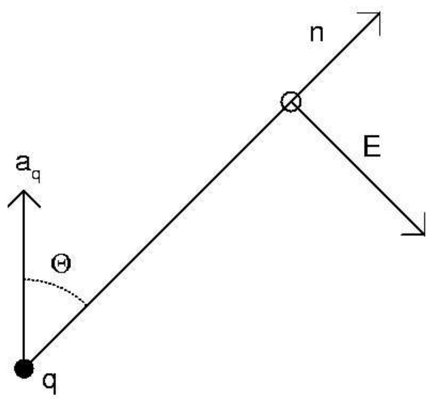

Figure 1: Electric field in the far zone of a non-relativistic accelerated point charge. ($\vec{B}$ is pointing out of the page).

Let the angle between the acceleration vector $\vec{a}_q$ and the line-of-sight vector $\hat{n}$ be Θ (as shown in Figure 1). The vector $\vec{E}$, from equation (63), must lie in the same plane as $\vec{a}_q$ and $\hat{n}$, but is perpendicular to $\hat{n}$. The vector $\vec{B}$, from equation (64), is perpendicular to both $\hat{n}$ and $\vec{E}$. The magnitude of both the $\vec{E}$ and $\vec{B}$ vector is

$$E = B = \frac{q}{c^2}\left[\frac{a_q}{R}\sin\Theta\right]_\tau \ . \tag{65}$$

If we take a charged particle initially at rest, and give it a little wiggle, the radiation field $\vec{E}_{\rm rad}$ and $\vec{B}_{\rm rad}$ created while the particle is accelerated will propagate outward at the speed of light, carrying energy along with it. Suppose that we are at location $\vec{r}$ and that the accelerated charged particle, at the time τ, was at location $\vec{r}_q(\tau)$, with speed $v_q(\tau) \ll c$, and acceleration $\vec{a}_q(\tau)$. The energy density of $\vec{E}$ and $\vec{B}$ at our location (assuming we're in the far zone) is

$$u = \frac{1}{8\pi}(E^2 + B^2) = \frac{1}{4\pi}\frac{q^2 a_q^2}{R^2 c^4}\sin^2\Theta \ , \tag{66}$$

where I have used the values of E and B from equation (65), in which Θ is the angle between our line of sight to the particle and the direction of its acceleration (both measured at the retarded time). If we hold up a tiny window, of area dA, perpendicular to the line-of-sight vector $\hat{n}$, the amount of energy flowing through the window in a short time dt will be

$$dE = ucdtdA \ . \tag{67}$$

The total energy flux (in ergs per square centimeter per second) is thus

$$F = \frac{dE}{dAdt} = u \cdot c = \frac{c}{4\pi} \frac{q^2 a_q^2}{R^2 c^4} \sin^2 \Theta \ . \tag{68}$$

Our tiny window (area dA) subtends a solid angle $R^2 d\Omega$ as seen from the particle's location, so the small amount of energy dE that flows through it in time dt can also be expressed as

$$dE = ucdtR^2 d\Omega \ , \tag{69}$$

and the power emitted by the charged particle per unit solid angle will be

$$\frac{dE}{d\Omega dt} = ucR^2 = FR^2 = \frac{q^2 a_q^2}{4\pi c^3} \sin^2 \Theta \ . \tag{70}$$

Thus, most of the power will be emitted in the direction nearly perpendicular to the charged particle's acceleration ($\Theta \approx \pi/2$); this angular distribution of power (illustrated in Figure 2) is called a dipole distribution. Consider, as

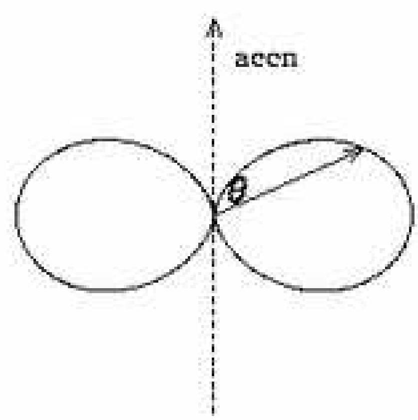

Figure 2: A dipole distribution of power ($dE/d\Omega dt \propto \sin^2 \Theta$).

an example, a charged particle moving in a circle at a constant speed. The acceleration vector will always point toward the circle's center, perpendicular to the particle's direction of motion. If you are viewing the particle's circular orbit face on, the line of sight will always be perpendicular to the acceleration vector, and you will always see the same flux.[6] If you see the orbit edge on, you will see the maximum flux when the particle is moving towards you or away from you, but you will see no flux when the particle is moving perpendicular to your line of sight. Integrated over all solid angles, the power emitted by the charged particle will be

$$P = \frac{dE}{dt} = \frac{q^2 a_q^2}{4\pi c^3} \int \sin^2 \Theta d\Omega = \frac{q^2 a_2^2}{2c^3} \int_0^\pi \sin^3 \Theta d\Theta \ . \tag{71}$$

The integral over Θ equals $4/3$, so we end up with a neat little formula for the power radiated by an accelerated charged particle:

$$P = \frac{2q^2 a_q^2}{3c^3} \ . \tag{72}$$

This useful little equation is called *Larmor's formula*, after the physicist Joseph Larmor.[7]

[6]As $\vec{a}_q$ sweeps through an entire circle, the electric field vector $\vec{E}$ will as well, and you will see circularly polarized radiation.

[7]Larmor was Senior Wrangler at Cambridge during the same year (1880) that J. J. Thomson was Second Wrangler. (Interestingly, William Thomson (later Lord Kelvin) had also been Second Wrangler; yet another confusing Thomson/Thomson similarity.)