1 Monday, October 17: Multi-particle Systems

For non-relativistic charged particles, we have derived a useful formula for the power radiated per unit solid angle in the form of electromagnetic radiation:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}[a_q^2 \sin^2\Theta]_\tau \ , \tag{1}$$

where q is the electric charge of the particle, $\vec{a}_q$ is its acceleration, and Θ is the angle between the acceleration vector $\vec{a}_q$ and the direction in which the radiation is emitted. The subscript τ is a quiet reminder that for any observer, we must use the values of a_q and Θ at the appropriate retarded time τ rather than the time of observation t. By integrating over all solid angles, we found the net power radiated by a non-relativistic charged particle:

$$P = \frac{2q^2}{3c^3}[a_q^2]_\tau \ . \tag{2}$$

Because the charge of the electron (or proton) is small in cgs units, and the speed of light is large in cgs units, we expect the power radiated by a single electron (or proton) to be small, even at the large accelerations that can be experienced by elementary particles.[1] The energy that the charged particle is radiating away has to come from somewhere. If the only energy source is the particle's kinetic energy, $E = mv_q^2/2$, the characteristic time scale for energy loss is

$$t_E = \frac{E}{P} = \frac{3c^3 m}{4q^2}\left[\frac{v_q^2}{a_q^2}\right]_\tau \ . \tag{3}$$

As a concrete example, consider an electron moving in a circle of radius r_q with a speed $v_q = \beta c$. The acceleration of the electron will be $a_q = \beta^2 c^2/r_q$, the power radiated will be

$$P = \frac{2q_e^2\beta^4 c}{3r_q^2} = 4.6\times 10^{-17}\,\mathrm{erg\,s^{-1}}\left(\frac{\beta}{0.01}\right)^4\left(\frac{r_q}{1\,\mathrm{cm}}\right)^{-2} \ . \tag{4}$$

[1] In problem set 2, the electron being bombarded by red light had a maximum acceleration of $a_q \sim 10^{19}\,\mathrm{cm\,s^{-2}}$; the proton in Lawrence's cyclotron had an acceleration of $a_q \sim 4\times 10^{16}\,\mathrm{cm\,s^{-2}}$.

The time scale for energy loss will be

$$t_E = \frac{3cm_e r_q^2}{4q_e^2\beta^2} = 8.9\times 10^5\,\mathrm{s}\left(\frac{\beta}{0.01}\right)^{-2}\left(\frac{r_q}{1\,\mathrm{cm}}\right)^2 . \tag{5}$$

It is informative to compare the time scale for energy loss with th orbital period of the electron:

$$t_P = \frac{2\pi r_q}{\beta c} = 2.1\times 10^{-8}\,\mathrm{s}\left(\frac{\beta}{0.01}\right)^{-1}\left(\frac{r_q}{1\,\mathrm{cm}}\right) . \tag{6}$$

For the electron's orbit to be stable, rather than a steep inward death spiral, we require $t_E \gg t_P$, which implies an orbital radius

$$r_q \gg \frac{8\pi q_e^2}{3c^2 m_e}\beta = \frac{8\pi r_0}{3}\beta , \tag{7}$$

where $r_0 \equiv q_e^2/(m_e c^2) = 2.8\times 10^{-13}\,\mathrm{cm}$ is the *classical electron radius.*

According to *classical* electromagnetic theory, then, an electron on an orbit smaller than $\sim \beta r_0$ in radius will radiate away its energy and spiral in to the origin on a time scale comparable to its orbital time. The radiation by electrons on circular orbits was what spelled the doom of classical atomic theory. Around 1911, Ernest Rutherford proposed a theory in which electrons went on circular orbits around a positively charged nucleus. The radius r of the orbits was determined by the requirement that the centripetal force be provided by the electrostatic force between the electron (charge q_e) and the nucleus (charge $-Zq_e$):

$$\frac{Zq_e^2}{r^2} = m_e\frac{\beta^2 c^2}{r} , \tag{8}$$

or

$$r = \frac{Zq_e^2}{\beta^2 m_e c^2} = \frac{Z}{\beta^2} r_0 . \tag{9}$$

However, it was known that the radius of the hydrogen atom ($Z = 1$) was $r \sim 0.5\,\text{Å} \sim 0.5\times 10^{-8}\,\mathrm{cm} \sim 2\times 10^4 r_0$. This implied $\beta = (r/r_0)^{-1/2} \sim 0.007$. So far, we're not in trouble; the electron orbit is large compared to the classical electron radius, and the speeds are non-relativistic. However, when we compute the time scale for energy loss from equation (5), we find

$$t_E = \frac{3cm_e r^2}{4q_e^2\beta^2} = \frac{3r_0}{4c}\left(\frac{r}{r_0}\right)^2\frac{1}{\beta^2} \sim 4\times 10^{-11}\,\mathrm{s} . \tag{10}$$

Thus, in the classical model, a typical atom should exist for less than a nanosecond. The durability of real atoms was explained by Neils Bohr in the context of quantum theory. In the Bohr model, electrons can only have orbits of specified energy, and can only emit or absorb photons that correspond to a difference in orbital energies. The moral of the story is that although Larmor's formula for the radiated power is very useful in some contexts (as applied to non-relativistic free electrons, for instance), it shouldn't be applied to electrons on bound orbits.

A single free electron, as noted above, usually doesn't emit much power. The luminous light sources we see in the universe around us contain many electrons, all radiating simultaneously. Suppose that some region in space contains N charged particles, which have trajectories $\vec{r}_i$ and charges q_i, where $i = 1, 2, \ldots, N$. Now, we could determine the electric and magnetic fields $\vec{E}_{\rm rad}$ and $\vec{B}_{\rm rad}$ at large distances by adding together the fields for each of the N particles. The practical difficulty to this approach is that for an observer at position $\vec{r}$ at time t, each of the N particles will have a different retarded time. Keeping track of the retarded times for a huge number of particles is an immense bookkeeping chore. It would be much easier if we could use the *same* retarded time for each particle. However, as my mom and dad always told me, the easy thing to do isn't always the right thing to do. Let's see under which circumstances we are permitted to use the same retarded time for each charged particle.

Suppose that the region we're looking at has a diameter $\sim L$; the light travel time across the region is then $t_L \sim L/c$. If the light travel time is much smaller than all time scales of interest in the problem, then we can justifiably make the approximation that the light we're observing from this region was all emitted simultaneously (and thus all has the same retarded time). If we are observing light at a wavelength λ, then one time scale of interest in the problem is the inverse frequency $\nu^{-1} = \lambda/c$. If $t_L \ll \nu^{-1}$ then the difference in phase between the front and back of the region can be ignored. This requirement implies

$$L/c \ll \lambda/c \ . \tag{11}$$

Thus, the requirement $L \ll \lambda$ is a necessary, but not sufficient, condition for using the same retarded time for all particles in the region. The other time scale of interest in the problem comes from looking at the motion of the charged particles. Each of the emitting particles is moving at a speed v_i, on an orbit of size $r_i \leq L$. Thus, the time it takes a particle to move a significant distance along its orbit is $t_i \sim r_i/v_i$. If $t_L \ll t_i$, then the distance a particle moves during a light crossing time is insignificant. This requirement implies

$$L/c \ll r_i/v_i \ , \tag{12}$$

or

$$v_i \ll c(r_i/L) \ll c \ . \tag{13}$$

Thus, if all motions are highly non-relativistic, and we are observing at a wavelength much longer than the size of the region we're looking at, we can make the approximation that all the particles in the region have the *same* retarded time.[2]

In general, for a region containing N charged particles, we can write

$$\vec{E}_{\rm rad} = \sum_i \frac{q_i}{c^2} \left[\frac{\hat{n} \times (\hat{n} \times \vec{a}_q)}{R_i} \right]_\tau \ , \tag{14}$$

where R_i is the distance from the observer to the $i^{\rm th}$ charged particle. If the observer is at a distance $R_0 \gg L$ from the system, then we can make the approximation that all the charged particles are at the same distance from the observer:

$$\vec{E}_{\rm rad} = \left[\frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R_0} \right]_\tau \ . \tag{15}$$

Here, I have used the *electric dipole moment* of the charged particles,

$$\vec{d} \equiv \sum_i q_i \vec{r}_i \ . \tag{16}$$

The dipole $\vec{d}$ used in equation (15) must be computed at the correct retarded time τ. However, as long as $\lambda \gg L$, and $v_i \ll c$ for all the charged particles, a single retarded time may be used for the entire ensemble of particles.

2 Wednesday, October 19: Fun with Dipoles

Suppose that we are looking at a collection of N charged particles contained within a region of maximum linear dimension L. Each of the particles has an

[2] You can just use the retarded time for a point that's stationary at the center of the observed region, for instance.

electric charge q_i and a trajectory $\vec{r}_i(t)$. We are at a location $\vec{r}$ well outside the distribution of charged particles, at a distance R_0 from the distribution's center. If the region containing the charges is small ($L \ll \lambda$), far away ($L \ll R_0$), and filled with non-relativistic particles ($v_i \ll c$), then we can express the electromagnetic potential at our location as a multipole expansion. The lowest order term in the resulting electric field strength involves the *electric dipole moment*. As we have seen,

$$\vec{E} \approx \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R_0} \ , \tag{17}$$

where $\hat{n}$ is the unit vector pointing from the charge distribution to us, and the electric dipole moment of the charged particles is the sum of their positions weighted by their electric charges:

$$\vec{d} \equiv \sum_{i=1}^{\infty} q_i \vec{r}_i \ . \tag{18}$$

With this *dipole approximation*, we can write the equation for the power per unit solid angle emitted by the N charged particles:

$$\frac{dP}{d\Omega} = \frac{|\ddot{\vec{d}}|^2}{4\pi c^3} \sin^2 \Theta \tag{19}$$

where Θ is the angle between $\ddot{\vec{d}}$ and the line-of-sight vector $\hat{n}$. The total power radiated is

$$P = \frac{2|\ddot{\vec{d}}|^2}{3c^3} \ . \tag{20}$$

As a simple example, consider an electric dipole that is constant in its orientation, but whose amplitude is sinusoidally oscillating:

$$\vec{d} = \hat{e}_d d_0 \cos(\omega_d t) \ . \tag{21}$$

[Engineering note: I gather this is basically what is happening in a dipole antenna. If you want to broadcast at a given radio frequency, you send a an alternating current at that frequency through a long thin bit of metal. The negatively charged electrons then slosh back and forth relative to the positively charged metal ions, causing an oscillating electric dipole.] Its second derivative is thus

$$\ddot{\vec{d}} = -\omega_d^2 \vec{d} \ . \tag{22}$$

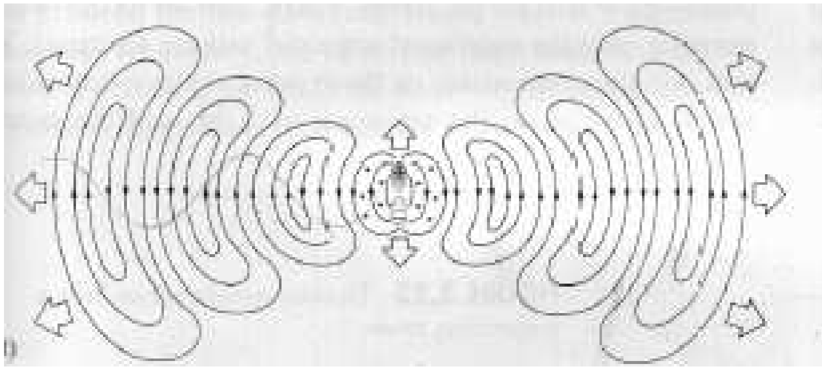

Figure 1: Sketch of $E_{\rm rad}$ for an oscillating dipole.

The electromagnetic field (see Figure 1) has the same frequency as the dipole oscillation:

$$E_{\rm rad} = \frac{\omega_d^2 d_0 \cos(\omega_d t)}{c^2 R_0} \sin\Theta \ . \tag{23}$$

The wavelength λ of the radiated electromagnetic field will be $\lambda = 2\pi c/\omega_d$: in other words, the wavelength of the emitted light is determined not by the physical size $\sim L$ of the distribution of charged particles, but by the oscillation frequency of its electric dipole.

The total power radiated by the oscillating dipole is

$$P = \frac{2\omega_d^4 d_0^2 \cos^2(\omega_d t)}{3c^3} \ . \tag{24}$$

Time-averaged over many oscillations, this becomes

$$\langle P \rangle = \frac{\omega_d^4 d_0^2}{3c^3} \ . \tag{25}$$

As an example, let's take the antenna of the radio station WOSU-AM. During the day, WOSU-AM broadcasts from a dipole antenna with a power of $\langle P \rangle = 5\,{\rm kilowatts} = 5 \times 10^{10}\,{\rm erg\,s^{-1}}$. The broadcast frequency is $\nu = 820\,{\rm kHz}$, requiring a dipole angular frequency of $\omega_d = 2\pi\nu = 5.15 \times 10^6\,{\rm s^{-1}}$. The magnitude of the antenna's dipole moment is then

$$d_0 = \left(\frac{3c^3 \langle P \rangle}{\omega_d^4} \right)^{1/2} \approx 8 \times 10^7\,{\rm esu\,cm} = 8 \times 10^{25}\,{\rm debye} \ . \tag{26}$$

The cgs unit of dipole strength, 1 debye $= 10^{-18}$ esu cm is tiny; it's the electric dipole you would get if you put an electron and proton 0.2 Å apart. This is because the debye been chosen to approximate the electric dipole of a single molecule with an asymmetric electron distribution.[3] You could produce a dipole of magnitude $d_0 = 8 \times 10^{25}$ debye by taking an electron and a proton and separating them by a distance $r = 1.6 \times 10^{17}$ cm $\approx 10^4$ AU (about a sixth of a light year). In practice, of course, it would be more practical to take Avogadro's number of protons and an equal number of electrons, and shift the electrons relative to the protons by a distance $r = 1.6 \times 10^{17}\,\text{cm}/6.0 \times 10^{23} \approx 3 \times 10^{-7}\,\text{cm} \approx 30$ Å.

It is my duty to inform you that the electric dipole approximation for the radiation field,

$$\vec{E}_{\rm rad} \approx \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R_0} \ , \tag{27}$$

is only the lowest order non-zero term in a complete multipole expansion. (The monopole term,

$$\vec{E} \approx \frac{\hat{n} \sum_i q_i}{R_0^2} \ , \tag{28}$$

is just a static Coulomb field, and doesn't represent radiation traveling away from the charged particles.) The next higher terms involve the *magnetic dipole moment*,

$$\vec{M} \equiv \frac{1}{2c} \sum_i q_i \vec{v}_i \times \vec{r}_i \ , \tag{29}$$

and the *electric quadrupole moment*,

$$\vec{D} \equiv \sum_i q_i [3\vec{r}_i(\hat{n} \cdot \vec{r}_i) - \hat{n} r_i^2] \ . \tag{30}$$

The magnetic dipole and electric quadrupole terms are, in general, smaller than the electric dipole term by a factor $\beta \sim v_i/c$. Thus, in non-relativistic systems, people only worry about the magnetic dipole and electric quadrupole terms when the electric dipole vanishes. For instance, in symmetric molecules (H_2 as compared to H_2O, for example) the electric dipole moment, measured relative to the molecule's center is zero. Molecular hydrogen is difficult to detect in the interstellar medium because it emits only (weak) electric quadrupole radiation, rather than the (stronger) electric dipole radiation produced by CO and other asymmetric molecules.

[3] A water molecule, for instance, has an electric dipole moment of 1.86 debye.

Anyway, that's all I want to say about the higher order multipole terms (at least for the moment). There are still lots of fun things we can do with electric dipoles, though. For instance, suppose that a linearly polarized electromagnetic wave is traveling along the x-axis, with

$$\vec{E} = \hat{e}_y E_0 \cos(kx - \omega t) \ . \tag{31}$$

If we place an electron at the origin, then we have shown, in problem set 2, that the electric force will make it undergo oscillations along the y axis, with

$$y(t) = \frac{q_e}{m_e}\frac{E_0}{\omega^2}(1 - \cos\omega t) \ . \tag{32}$$

(Remember, also, that this results only holds true when the electron's motions are non-relativistic.) The dipole moment associated with the electron,

$$\vec{d}(t) = \hat{e}_y q_e y(t) \ , \tag{33}$$

thus has a second derivative in time of

$$\ddot{\vec{d}}(t) = \hat{e}_y q_e \ddot{y} = \hat{e}_y \frac{q_e^2 E_0}{m_e} \cos\omega t \ . \tag{34}$$

Thus, the electron, which is being accelerated by the light falling upon it, is emitting light (of the same frequency) because of its acceleration. The power the electron emits, per unit solid angle, is

$$\frac{dP}{d\Omega} = \frac{|\ddot{\vec{d}}|^2}{4\pi c^3}\sin^2\Theta = \frac{q_e^4 E_0^2}{4\pi m_e^2 c^3}\cos^2\omega t \sin^2\Theta \ . \tag{35}$$

Taking the time average over many oscillations, and using the definition of the classical electron radius, $r_0 \equiv q_e^2/(m_e c^2)$, we may write

$$\frac{dP}{d\Omega} = \frac{cE_0^2 r_0^2}{8\pi}\sin^2\Theta \ . \tag{36}$$

The total power radiated by the electron, integrated over all directions, is

$$P = \frac{cE_0^2 r_0^2}{3} \ . \tag{37}$$

The electron is taking some of the power from the plane wave striking it, and re-emitting it in a dipole distribution (mostly perpendicular to the dipole's oscillations along the y axis). Since the re-emission is instantaneous, this is more easily thought of as a type of scattering, rather than as true absorption. In fact, this interaction is the phenomenon we referred to as "Thomson scattering" on Friday, September 23, the second day of class.[4] The energy density of the plane wave striking the electron is

$$u = \frac{E_0^2}{8\pi} \ , \tag{38}$$

and the energy flux (power per unit area) past the electron's location is

$$F = uc = \frac{cE_0^2}{8\pi} \ . \tag{39}$$

Thus, we can determine the effective cross-section of the electron by the simple calculation

$$\sigma = \frac{P}{F} = \frac{cE_0^2 r_0^2}{3} \frac{8\pi}{cE_0^2} = \frac{8\pi}{3} r_0^2 = 6.65 \times 10^{-25}\,\mathrm{cm}^2 \ . \tag{40}$$

This cross-section for scattering of light by a non-relativistic electron is known as the *Thomson cross-section*, and is usually designated by the symbol σ_T. Note that although the Thomson cross-section is independent of the frequency of light, the non-relativistic assumption breaks down for very high intensity light, and the classical assumption breaks down for very high frequency light ($h\nu \geq m_e c^2 \sim 0.5\,\mathrm{MeV}$). When $h\nu \geq m_e c^2$, the resulting scattering (called "Compton scattering") must be dealt with quantum mechanically.

3 Friday, October 21: Scattering by Electrons

Linearly polarized light, with

$$\vec{E} = \hat{e}_y E_0 \cos(kx - \omega t) \ , \tag{41}$$

is scattered by a non-relativistic electron, with cross-section

$$\sigma_T = \frac{1}{F} P = \frac{8\pi}{3} r_0^2 \ , \tag{42}$$

[4] Calculating the results of Thomson scattering is a good example of a situation where it's useful to think of light as a wave, rather than a particle.

where F is the flux of incident light, and P is the power radiated by the oscillating electron. The scattering is anisotropic; by analogy with the total cross-section σ, we can define a differential cross-section $d\sigma$ for scattering into the small solid angle $d\Omega$:

$$\frac{d\sigma}{d\Omega} = \frac{1}{F}\frac{dP}{d\Omega} \ . \tag{43}$$

From our knowledge of F and $dP/d\Omega$,[5] we can write

$$\frac{d\sigma}{d\Omega} = \frac{8\pi}{cE_0^2}\frac{cE_0^2 r_0^2}{8\pi}\sin^2\Theta = r_0^2\sin^2\Theta \ , \tag{44}$$

where Θ is the angle between the direction of of motion for the scattered light ($\hat{n}$) and the direction of oscillation of the electron (which in turn is equal to $\hat{e}_y$, the direction of $\vec{E}$ for the incident light). Linearly polarized light that is scattered by an electron is also linearly polarized; its plane of polarization is the plane defined by $\hat{n}$ and $\hat{e}_y$.[6]

The above analysis considered the scattering of linearly polarized light. But what happens when the incident light is *unpolarized*? Let's suppose that unpolarized light is propagating along the x axis, and encounters an electron at the origin. We want to know the differential scattering cross-section $d\Sigma/d\Omega$ for light scattered at an angle ϑ. That is, $\cos\vartheta = \hat{n}\cdot\hat{e}_x$, so that light scattered in the same direction as the incident light has $\vartheta = 0$, light scattered in the opposite direction has $\vartheta = \pi$, and light scattered at right angles to the incident light has $\vartheta = \pi/2$. If the light is unpolarized, we expect the results to possess rotational symmetry about the x axis, so the differential cross-section should be a function of ϑ alone.

Finding the exact result for the differential cross-section requires some three-dimensional geometry, which is a bit tricky to visualize if you are geometrically challenged, as I am. I will content myself with writing down the result:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}r_0^2(1+\cos^2\vartheta) \ , \tag{45}$$

[5]See the previous lecture.

[6]Since $\sigma_T \propto r_0^2$ and $r_0 \propto 1/m_e$, we can compute that the scattering cross-section for the proton is smaller than that for the electron by a factor $(m_e/m_p)^2 \approx 3\times 10^{-7}$; this is why we only talk about *electron* scattering in an astrophysical plasma, and ignore scattering by protons and helium nuclei.

where ϑ is the angle between the directions of propagation of the incident (unpolarized) light and the scattered light. Note that the scattered power is greatest at $\vartheta = 0$ and $\vartheta = \pi$ (straight ahead and straight behind), and is minimized, but still greater than zero, at $\vartheta = \pi/2$ (off to the side). The total cross-section σ for unpolarized light, if you integrate over solid angle, is the same as that for unpolarized light. The degree of polarization of the scattered light depends strongly on ϑ. If you are an observer on the x axis, you will see the electron being accelerated hither and yon on the $x = 0$ plane, as $\vec{E}$ fluctuates in angle and amplitude. The scattered light seen from along the x axis ($\vartheta = 0$ and $\vartheta = \pi$) will be unpolarized as a result, since there is no preferred direction of motion for the electron in the $x = 0$ plane. Now, however, suppose you are an observer along the y axis (where $\vartheta = \pi/2$). The component of the electron's acceleration along the y axis won't produce any radiation at your location; only the acceleration along the z axis will. Thus, you will see *linearly polarized* light, with $\vec{E}$ lying perpendicular to the direction of propagation of the incident light (see Figure 2). In general, the

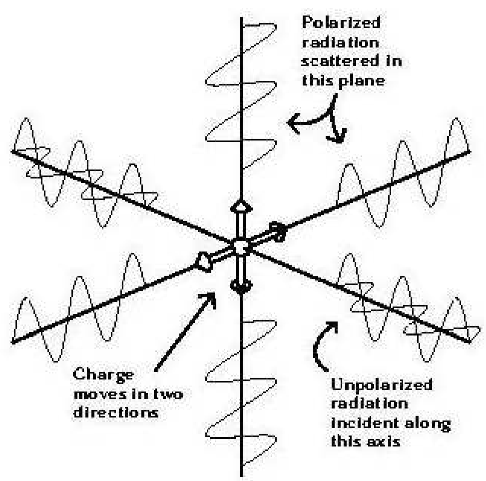

Figure 2: Thomson scattering of unpolarized light.

degree of polarization of the scattered light is

$$\Pi = \frac{1 - \cos^2 \vartheta}{1 + \cos^2 \vartheta} \ . \tag{46}$$

Thus, even if light is born unpolarized, it can have polarization thrust upon it by being scattered at right angles by an electron.

If you hang out with cosmologists, you will occasionally hear them talk about the polarization of the Cosmic Microwave Background (CMB). If the CMB had a perfect blackbody spectrum, then it would be totally unpolarized. However, as the light of the CMB moves toward us, it can acquire polarization by Thomson scattering from electrons. Suppose that an electron is located in intergalactic space. Very far away from the photon, along the z axis, an observer is located (Figure 3). Unpolarized light from the CMB approaches

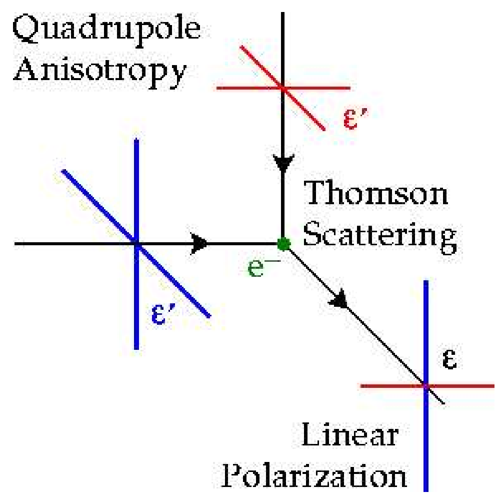

Figure 3: Polarization of the Cosmic Microwave Background. Blue lines represent hotter radiation; red lines represent cooler radiation.

the electron along the x axis; the light scattered along the z axis is linearly polarized in the y direction. Of course, there is also unpolarized light from the CMB approaching the electron along the y axis; when this light is scattered along the z axis, it is polarized in the x direction. If the CMB were perfectly isotropic, then the power of the two scattered beams would be identical, and the orthogonally polarized beams would add together to make unpolarized light. However, the CMB isn't exactly isotropic. Suppose the CMB light traveling along the x axis were slightly hotter (and hence had a slightly higher flux) then the CMB light traveling along the y axis. In that case, the scattered light from the two orthogonally fluxes would be slightly unequal in power, and there would be a (very small) net polarization in the y direction. Thus, if we see temperature fluctuations on large angular scales in the CMB – and we do – we should also see a small amount of polarization due to Thomson scattering from intergalactic electrons – and we do.

When we considered Thomson scattering, we stated that the only force on the electron was the electric force from the incident light, so that

$$\frac{d^2y}{dt^2} = \frac{q_e}{m_e} E_0 \cos\omega t \ . \tag{47}$$

However, there may be other significant forces at work on the electron. The radiated energy, for instance, is drawn from the kinetic energy of the electron, so that while the electron is *driven* by the electric field, it is *damped* by the loss of energy in the form of radiation. As we saw on Monday, the time scale for energy loss from an electron is

$$t_E = \frac{3m_e c^3}{4q_e^2}\frac{v_q^2}{a_q^2} = \frac{3c}{4r_0}\frac{v_q^2}{a_q^2} \ , \tag{48}$$

For a sinusoidally oscillating particle, the mean square velocity divided by the mean square acceleration is $\langle v_q^2\rangle/\langle a_q^2\rangle = \omega^{-2}$, so the time scale for energy loss, averaged over many oscillations, is

$$t_E = \frac{3c}{4r_0\omega^2} \ . \tag{49}$$

The damping is negligible when the time scale for damping, t_e, is much longer than the time for one oscillation, $t_P = 2\pi/\omega$. This criterion is satisfied when

$$\frac{3c}{4r_0\omega^2} \gg \frac{2\pi}{\omega} \ , \tag{50}$$

or

$$\omega \ll \frac{3}{8\pi}\frac{c}{r_0} \approx 10^{22}\,\mathrm{s}^{-1} \ , \tag{51}$$

corresponding to emitted photon energies $\hbar\omega \ll 8\,\mathrm{MeV}$. Thus, as long as we are in the correct energy regime for Thomson scattering (photon energy less than 0.5 MeV), we can ignore the effects of damping.

Let's examine another case when there may be an additional force acting on the electron. Suppose that the electron is not free, but is bound in an oscillator with natural frequency ω_0. In the absence of any light, the equation of motion for the bound particle would be

$$\frac{d^2y}{dt^2} = -\omega_0^2 y(t) \ , \tag{52}$$

implying a natural (undriven) oscillation of the form

$$y \propto \cos(\omega_0 t + \phi) \ . \tag{53}$$

If the bound oscillator is bombarded with monochromatic light of frequency $\omega \neq \omega_0$, the equation of motion for the particle will be

$$\frac{d^2 y}{dt^2} = -\omega_0^2 y + \frac{q_e}{m_e} E_0 \cos \omega t \ . \tag{54}$$

You can demonstrate by substitution that a solution to this equation is

$$y(t) = y_0 \cos \omega t \ , \tag{55}$$

where the amplitude of oscillations is

$$y_0 = -\frac{q_e E_0}{m_e} \frac{1}{\omega^2 - \omega_0^2} \ . \tag{56}$$

The frequency of the electron's driven oscillations are unaffected by the electron's being bound; however, the amplitude of the oscillations is altered. The total power radiated by this oscillating dipole is

$$P = \frac{q_e^2 y_0^2 \omega^4}{3c^3} = \frac{q_e^4 E_0^2}{3 m_e^2 c^3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2} = \frac{c r_0^2 E_0^2}{3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2} \ . \tag{57}$$

Note that near the resonance, $\omega \approx \omega_0$, the power emitted is huge; in this case, the damping time t_E becomes very short, and the damping term in the equation of motion can not be ignored. When the frequency is not near the resonance, the cross-section of the bound oscillator is

$$\sigma = \frac{P}{F} = \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2} \ . \tag{58}$$

At high frequencies, $\omega \gg \omega_0$, this just reduces to the Thomson cross-section. At low frequencies, however, where $\omega \ll \omega_0$, the cross-section is strongly dependent on frequency:

$$\sigma \approx \sigma_T \left(\frac{\omega}{\omega_0} \right)^4 \ . \tag{59}$$

Scattering in this regime is called *Rayleigh scattering*. Oxygen and nitrogen molecules have a resonance frequency of $\omega_0 \sim 6 \times 10^{15}\,\mathrm{s}^{-1}$ for the electrons in their outer energy levels (this corresponds to $\lambda_0 \sim 300\,\mathrm{nm}$, in the ultraviolet). Thus, at frequencies lower than ultraviolet frequencies, the molecules in the Earth's atmosphere demonstrate Rayleigh scattering, with the higher frequency visible light (violet and blue) being more strongly scattered than the lower frequency light (orange and red). And that, boys and girls, is why the sky is blue.