1 Monday, October 24: Special Relativity Review

Personally, I'm very fond of classical physics. As we've seen recently, you can derive some very useful electromagnetic formulae without taking into account quantum mechanics or special relativity. However, just as Max Planck demonstrated that you sometimes have to take quantization into effect, Albert Einstein (just a century ago) demonstrated that you sometimes have to take relativistic effects into account. There are times when classical Newtonian physics is an inadequate approximation to reality. In this course, I'm trying to minimize the use of quantum mechanics, for fear of intruding into the domain of your other radiation course (Astronomy 823: Theoretical Spectroscopy). However, I am going to plunge into special relativity, to see how our previously derived results vary in the limit that the speed of charged particles approaches the speed of light.

One of the more entertaining aspects of the universe, at present, is its great variety. There's a huge range of densities, temperatures, and electric and magnetic field strengths. In a few regions of the universe, the temperature is high enough for electrons to be relativistic ($v_e \sim c$). For electrons to have thermal velocities near the speed of light, the thermal energy must be comparable to or greater than the rest energy of the electron:

$$kT \geq m_e c^2 \sim 0.5\,\mathrm{MeV}\ , \tag{1}$$

which requires a temperature[1]

$$T \geq m_e c^2/k \sim 6 \times 10^9\,\mathrm{K}\ . \tag{2}$$

In a magnetic field of flux density B, an electron on an orbit of radius r will be relativistic if

$$\omega_{\mathrm{cyc}} r = \left|\frac{q_e B}{m_e c}\right| r \sim c\ . \tag{3}$$

This requires

$$|B| r \sim \frac{m_e c^2}{|q_e|} \sim 2000\,\mathrm{gauss\,cm}\ . \tag{4}$$

Near a magnetized neutron star with $B \sim 10^9$ gauss, even electrons moving in tiny orbits $r \sim 20\,\mathrm{nm}$ require relativistic treatment.

[1] Generally useful approximation: $1\,\mathrm{MeV} \to 10^{10}\,\mathrm{K}$.

The special theory of relativity (alias *special relativity*) is based on two simple postulates. The first postulate states:

> (1) The laws of physics are the same in all inertial frames of reference.

An inertial frame of reference is one in which Newton's Laws of Motion hold true. Thus, in an inertial frame,

$$m\ddot{\vec{r}} = \vec{F} \ , \tag{5}$$

where m is a particle's mass and $\vec{F}$ is the net force on the particle. A *rotating* frame of reference is an example of a frame that is *not* inertial. In a frame rotating with a constant angular velocity $\vec{\Omega}$, the equation of motion is

$$m\ddot{\vec{r}} = \vec{F} - m(2\vec{\Omega} \times \dot{\vec{r}}) - m(\vec{\Omega} \times [\vec{\Omega} \times \vec{r}]) \ . \tag{6}$$

Thus, in a rotating frame of reference, there are two fictitious forces; the "Coriolis force" (proportional to $\vec{\Omega} \times \dot{\vec{r}}$) and the "centrifugal force" (proportional to $\vec{\Omega} \times [\vec{\Omega} \times \vec{r}]$). An inertial frame may also be defined as a frame in which there are no fictitious forces.

Any frame that is moving at a constant velocity $\vec{v}$ with respect to an inertial frame of reference is also inertial. Thus, Einstein's first postulate is telling us that the laws of reference are the same in two inertial frames, no matter what their relative velocity $\vec{v}$. In fact, this postulate long predates Einstein. In his book "Dialogue Concerning the Two Chief World Systems", Galileo pointed out that the laws of motion on a sailing ship moving a constant speed were the same as those on dry land. The postulate that led to the mind-bendingly new results of special relativity was Einstein's second postulate:

> (2) The speed of light in a vacuum (c) is the same in all inertial frames of reference.

Thus, Einstein implies that the statement $c \equiv 299{,}972{,}458\,\mathrm{cm\,s^{-1}}$ is a law of physics, independent of which inertial reference frame you choose. To see some of the implications of Einstein's seemingly innocuous postulates, consider two inertial reference frames, K and K' (Figure 1). The origins of the two frames coincide at $t = 0$, but the frame K' is moving along the x axis with respect to the frame K. The velocity of K' with respect to K is

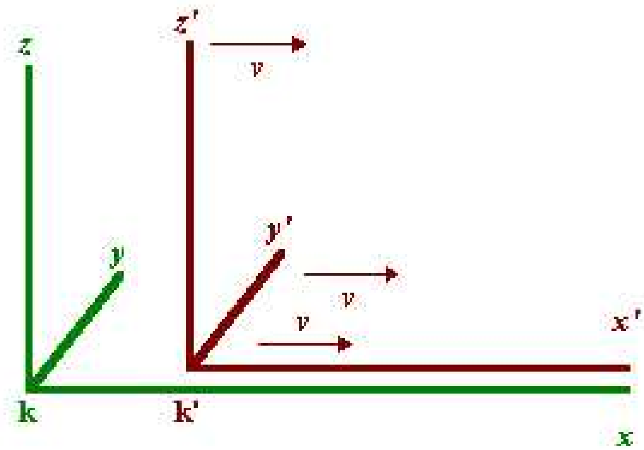

Figure 1: Two inertial frames of reference, with relative velocity v.

$\vec{v} = v\hat{e}_x$. At the moment $t = 0$, a brief pulse of light is emitted at the origin. An observer in the K frame sees the shell of light expand outward with speed c. Thus, in the K frame, the equation of the expanding spherical shell is

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \ . \tag{7}$$

However, in the K' frame, an observer *also* sees the shell of light expanding outward with speed c. Thus, in the K' frame,

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \ . \tag{8}$$

Notice how subtly I have introduced the notion that t is not necessarily equal to t'; that clocks can run at different rates in different reference frames. The two equations (7 and 8) can be satisfied if the coordinates in the two frames are related by the *Lorentz transformation*:

$$\begin{aligned} x' &= \gamma(x - vt) & (9) \\ y' &= y & (10) \\ z' &= z & (11) \\ t' &= \gamma(t - vx/c^2) \ , & (12) \end{aligned}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \ . \tag{13}$$

Thus, the γ factor approaches infinity as v approaches c. The inverse of the Lorentz transformation is

$$\begin{aligned} x &= \gamma(x' + vt') & (14) \\ y &= y' & (15) \\ z &= z' & (16) \\ t &= \gamma(t' + vx'c^2) \ . & (17) \end{aligned}$$

This follows straightforwardly; if K' is moving at a speed v with respect to K, then K is moving at a speed $-v$ with respect to K'. In general, an event that occurs at space-time coordinates (x, y, z, t) in the K frame will occur at a different space-time coordinates (x', y', z', t') in the K' frame. The inability of the two observers to agree on a time t or a location $\vec{r}$ for a given event leads to amusing relativistic effects such as length contraction (sometimes called Lorentz-Fitzgerald contraction) and time dilation.

Since distances and lengths as measured in the two frames differ, particle velocities as measured in the two frames must also differ. From the Lorentz transformation,

$$\begin{aligned} u_x = \frac{dx}{dt} &= \frac{\gamma(dx' + vdt')}{\gamma(dt' + vdx'/c^2)} = \frac{u'_x + v}{1 + vu'_x/c^2} & (18) \\ u_y = \frac{dy}{dt} &= \frac{u'_y}{\gamma(1 + vu'_x/c^2)} & (19) \\ u_z = \frac{dz}{dt} &= \frac{u'_z}{\gamma(1 + vu'_x/c^2)} \ . & (20) \end{aligned}$$

Note the similar form for u_y and u_z, representing motion perpendicular to $\vec{v}$. This indicates that it might be useful to divide $\vec{u}$ (the velocity measured in the K frame) into a component parallel to $\vec{v}$ and a component perpendicular to $\vec{v}$. With this decomposition,

$$u_\parallel = u_x = \frac{u'_\parallel + v}{1 + vu'_\parallel/c^2} \quad (21)$$

and

$$u_\perp = (u_y^2 + u_z^2)^{1/2} = \frac{u'_\perp}{\gamma(1 + vu'_\parallel/c^2)} \ . \quad (22)$$

This implies that the direction of motion of a particle will differ in the two inertial frames. Suppose that in the K' frame, the particle is moving at an angle θ' relative to $\vec{v}$, so that $u'_\parallel = u' \cos\theta'$ and $u'_\perp = u' \sin\theta'$. In the K frame, the particle will be moving at an angle θ relative to $\vec{v}$, where θ is given by the relation

$$\tan\theta = \frac{u_\perp}{u_\parallel} = \frac{u' \sin\theta'}{\gamma(u' \cos\theta' + v)} \ . \tag{23}$$

Although $\theta' = 0$ implies $\theta = 0$, in general you have $\tan\theta \neq \tan\theta'$ for $v > 0$.

It is illuminating to consider the case of a particle that is moving perpendicular to the $\vec{v}$ axis, as seen by the K' observer. That is, $u'_\perp = u'$, $u'_\parallel = 0$, and $\theta' = \pi/2$. As seen by the K observer, the velocity components of the particle are

$$u_\parallel = v \tag{24}$$

and

$$u_\perp = u'/\gamma \ . \tag{25}$$

Thus, the velocity perpendicular to $\vec{v}$ is decreased, and the velocity parallel to $\vec{v}$ is increased. This effect is known as *relativistic beaming*, and applies to all particles, even photons. For a photon, $u' = c$ (of course), and a photon emerging at $\theta' = \pi/2$ in the K' frame emerges at an angle

$$\tan\theta = \frac{u_\perp}{u_\parallel} \approx \frac{c/\gamma}{v} \tag{26}$$

in the K frame. In the case of highly relativistic motion of K relative to K', the value of θ reduces to

$$\theta \approx \tan\theta \approx \frac{1}{\gamma} \ll 1 \ . \tag{27}$$

If radiation is being emitted isotropically in the K' frame, half will be emitted with $\tan\theta' < \pi/2$. In the K frame, all these photons will be crammed into the tiny cone with $\theta < 1/\gamma$.

2 Wednesday, October 26: Electromagnetic Transformations

There exists an extremely powerful and elegant mathematical formalism for dealing with Lorentz transformations. Sections 4.2 and 4.3 of the textbook go into loving detail, discussing covariant four-vectors, contravariant four-vectors, second-rank tensors (symmetric and antisymmetric), and, in general, all the mathematical apparatus that makes relativistic calculations so compact in appearance. Since time is short, and I am less interested in mathematical elegance than in rough-and-ready physics, I am going to skip over the mathematically sophisticated parts of chapter 4, and plunge right into the transformation of Maxwell's equations in going from one inertial frame K to another inertial frame K'.

Einstein's first postulate stated that the laws of physics are the same in all inertial frames of reference. Among those laws are Maxwell's equations. In an inertial frame K, Maxwell's equations in a vacuum are

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{28}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{29}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{30}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \ . \tag{31}$$

Maxwell's equations must also hold true in a different inertial frame K', moving at a velocity $\vec{v}$ with respect to K. In both frames, the speed of light c will be the same. However, the spatial derivatives and time derivatives will be different in the two frames; as a result, we expect $\vec{E}'$ to differ from $\vec{E}$ and $\vec{B}'$ to differ from $\vec{B}$.

Rybicki and Lightman, using the full power of tensor analysis, derive the transformation from $\vec{E}$ to $\vec{E}'$ and from $\vec{B}$ to $\vec{B}'$. In the *Feynman Lectures on Physics* (volume II, chapter 26), Feynman computes the Lorentz transform of $\vec{E}$ and $\vec{B}$. He says, "It isn't at all difficult to do; it's just laborious – the brains involved are nil, but the work is not." In other words, the computation is straightforward but tedious, and I will not grind through all the steps.

What I am going to do is to look at very simple physical models from which we can derive the transformations. First, consider a simple capacitor (Figure 2), in which two circular plates of diameter D are separated by a distance $d \ll D$. One plate has a charge $+q$, and the other has a charge $-q$. In an inertial frame K where the capacitor is at rest, the electric field between the plates is

$$E = 4\pi\Sigma \ , \tag{32}$$

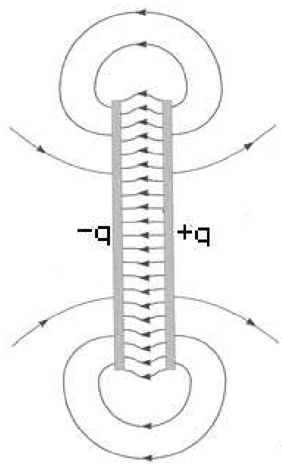

Figure 2: A capacitor: $\vec{E} \approx$ constant between plates.

where $\Sigma = q/(\pi D^2)$ is the charge density of the positively charged plate. This is one of the basic results of electrostatics, and probably has triggered a flashback to your sophomore physics class. The direction of $\vec{E}$ is perpendicular to the plates, and points from the positively charged plate to the negatively charged plate. (Note that as long as $d \ll D$, the electric field between the plates is independent of the plate separation d.) The magnetic flux density B is zero, since there is no current as measured in the K frame.

Consider an inertial frame K' moving perpendicular to the capacitor plates, and thus *parallel* to $\vec{E}$, at a speed $\vec{v} = \vec{\beta}c$. In this frame, the area of each plate is unchanged (since length contraction only occurs in the direction of $\vec{\beta}$). Since electric charge q is also unaffected by a switch in reference frames,[2] the charge density $\Sigma' = \Sigma$ is unchanged. Although the distance between the plates is decreased by length contraction ($d' = d/\gamma$), the electric field is independent of the the separation between plates. Thus, we learn that

$$E'_{\parallel} = E_{\parallel} \ . \tag{33}$$

The electric field is unchanged if it's parallel to the direction of motion $\vec{\beta}$.

Now consider an inertial frame K' moving parallel to the capacitor plates, and thus *perpendicular* to $\vec{E}$, at a speed $\vec{v} = \vec{\beta}c$. In this frame, the circular plates are contracted to ellipses, with long axis D and short axis D/γ. Since

[2]The invariance of electric charge is observationally verified to high precision.

the charge of each plate is still unaffected, the charge density is increased to $\Sigma' = \gamma\Sigma$. Thus, we learn that

$$E'_{\perp} = 4\pi\Sigma' = \gamma(4\pi\Sigma) = \gamma E_{\perp} \ . \tag{34}$$

Thus, the electric field is increased by a factor of γ if it's perpendicular to the direction of motion $\vec{\beta}$ of the inertial frame K'. In addition, there will now be a magnetic field B' between the two plates. Consider yourself as an observer in the K' frame passing between the two plates of the capacitor with a speed βc. From your point of view, you will see a current sheet of surface current density $\mu' = \Sigma'\beta c$ on one side of you as the negatively charged plate slips past with speed $-\beta c$, and another of surface current density $-\mu' = -\Sigma'\beta c$ on the other side as the positively charged plates slips past. Thus, you are between two current sheets, effectively infinite in size when $d \ll D/\gamma$. The solution to this problem is one of the basic results of magnetostatics: the magnetic field between the current sheets is perpendicular to $-\vec{\beta}c$, the direction of the current, and has a magnitude

$$B' = \frac{4\pi}{c}\mu' = 4\pi\Sigma'\beta = \beta E'_{\perp} \ . \tag{35}$$

Since the generated magnetic field is perpendicular to both $\vec{\beta}$ and $\vec{E}_{\perp}$, we may write

$$B'_{\perp} = -\vec{\beta} \times \vec{E}'_{\perp} = -\gamma\vec{\beta} \times \vec{E}_{\perp} \ . \tag{36}$$

By considering a capacitor in motion, we considered a case in which the only field in the K frame (the frame in which the capacitor was motionless) was a constant electric field $\vec{E}$. In this way, we found how $\vec{E}$ is transformed to $\vec{E}'$ (and $\vec{B}'$) as seen from an inertial frame moving parallel or perpendicular to $\vec{E}$. By symmetry, to see how the magnetic flux density $\vec{B}$ transforms, we should consider a case in which the only field in the K frame is a constant magnetic field $\vec{B}$. We can produce a constant $\vec{B}$ by setting up two current sheets, one of surface current density $\vec{\mu}$, and the other of surface current density $-\vec{\mu}$, separated by a distance d that is small compared to the width D of the current sheets. In each sheet, the current is created by a surface charge density $-\Sigma_e$ of electrons moving with a drift velocity $\pm\vec{u}_e$ relative to a surface charge density $+\Sigma_e$ of stationary, positively charged ions. The net surface charge vanishes in the K frame, so $\vec{E} = 0$ in that frame. The magnetic flux density in the K frame has the magnitude

$$B = \frac{4\pi}{c}\mu = \frac{4\pi}{c}\Sigma_e u_e \ , \tag{37}$$

and is oriented perpendicular to the electron velocity $\pm\vec{u}_e$, and parallel to the current sheets. (This is the same familiar "infinite current sheet" result that I invoked earlier in the lecture.)

Consider an inertial frame K' moving parallel to $\vec{B}$, and thus perpendicular to the electron drift velocity $\pm\vec{u}_e$. In this frame, the width of the current sheets is contracted by and amount $D' = D/\gamma$; thus, the surface density of electrons is increased to $\Sigma'_e = \gamma\Sigma_e$. Thanks to the results of time dilation, the electron drift velocity is decreased to $u'_e = u_e/\gamma$. Thus, the effects of time dilation and length contraction exactly cancel:

$$B'_{\parallel} = \frac{4\pi}{c}\Sigma'_e u'_e = \frac{4\pi}{c}(\gamma\Sigma_e)(u_e/\gamma) = B_{\parallel} \ . \tag{38}$$

Just as $E_{\parallel}$ is unchanged by the the switch of inertial frames, $B_{\parallel}$ is also unchanged.

To see how $B_{\perp}$ is changed by the switching inertial frames, we need to place ourselves in a frame of reference K' that is moving perpendicular to $\vec{B}$ and parallel to $\pm\vec{u}_e$. The complete calculation in this case requires computing the drift velocities u'_e of the electrons in both sheets and the drift velocities u'_i of the positive ions in both sheets. Then the surface current density in each sheet is calculated, and the resulting $B'_{\perp}$ is calculated. This is a bit complicated, algebraically speaking; I, like Rybicki and Lightman, will leave it as an exercise for the reader, merely quoting the results:

$$B'_{\perp} = \gamma B_{\perp} \ , \tag{39}$$

and

$$\vec{E}'_{\perp} = \vec{\beta} \times \vec{B}'_{\perp} = \gamma\vec{\beta} \times \vec{B}_{\perp} \ . \tag{40}$$

Note that an electrically neutral current sheet, with $\vec{E} = 0$, shows an electric field when you move past it in a direction perpendicular to $\vec{B}$. This is because the electrons and protons are moving with a different velocities parallel to $\vec{\beta}$, and thus show different amounts of length contraction.

Adding together all the terms that we have derived from moving capacitors and current sheets around, we find that

$$\begin{aligned}
\vec{E}'_{\parallel} &= \vec{E}_{\parallel} && (41)\\
\vec{E}'_{\perp} &= \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B}) && (42)\\
\vec{B}'_{\parallel} &= \vec{B}_{\parallel} && (43)\\
\vec{B}'_{\perp} &= \gamma(\vec{B}_{\perp} - \vec{\beta} \times \vec{E}) \ . && (44)
\end{aligned}$$